Handout 3 – Supplement

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Time invariant system: A system is time invariant if applying a time shift k to the input of a system, $x(n-k)$, and then running it through a transformation (a system), $T[x(n-k)]$, is identical to applying the original input through the system, $T[x(n)]$, and then shifting, $\{T[x]\}(n-k)$. If this relation holds for every k and every input signal $x(n)$, then we call the system T time invatiant.

Time variant system: A system that is not time invariant is called time variant. As an example, consider the system $y(n) = nx(n)$. Suppose that the input is an impulse, $x(n) = \delta(n)$. Let us begin by applying the system to the input, $y(n) = nx(n)$, we see that $y(n=0) = nx(n=0) = 0 \cdot 1 = 0$. Similarly, for all $n \neq 0$ we have $x(n) = 0$, and so $y(n) = 0$. We see that y is zero for all n. Shifting $y(n)$ by any k results in an all-zero signal.

Next, suppose that we shift $x(n)$ by k. That is, $x(n-k) = \delta(n-k)$, which will be 1 when $n = k$. But $y(x(n-k))$ will then be k (and not zero). In words, taking the signal and running it through the system yields an all zero output, and all its shifts are all zero; but shifting the signal and then running it through the system yields an output that has a non-zero value at time $n = k$.

Unstable system: We have defined a bounded input bounded output (BIBO) system as one where for every bounded input $x(n)$, the output $y(n)$ will also be bounded. Let us consider an example of an unstable system. Define a prototype system with feedback,

$$y(n) = \alpha y(n-1) + x(n), \tag{1}$$

where α is a constant. Larger α means that the past will have a greater say in determining future values of the output signal.

Example 1: To keep things simple, suppose that $\alpha = 1$, meaning that $y(n) = y(n-1)+x(n)$, the output is the sum of its previous value and the present input. Suppose further that our input is a step function, $x(n) = u(n)$, where we remind the reader than $u(n) = 1$ for $n \geq 0$, else $u(n) = 0$ for $n < 0$. Let us now compute $y(n)$ by simulating it. Suppose that for negative n the output started at zero. Then at time $n = 0$ we have

$$y(n=0) = y(n-1) + x(n) = y(n=-1) + x(0) = 0 + 1 = 1.$$

Next, we consider $n = 1$,

$$y(n=1) = y(n-1) + x(n) = y(n=0) + x(1) = 1 + 1 = 2.$$

It can be shown that for $n > 0$ we will have $y(n) = n + 1$. This can be shown, for example, using a recursion argument. We see that the input (the step function) is bounded, and yet the output is unbounded, and its value grows as n increases. Therefore, this system is not BIBO-stable.

Example 2 (counter-example): For the step function input, $x(n) = u(n)$, we have seen that the output is unbounded. In contrast, if the input is the impulse, $x(n) = \delta(n)$, then things change. For $n \leq 0$, $\delta(n) = u(n)$, and so $y(n) = 0$ for $n < 0$, and $y(n = 0) = 1$. Things change starting from time $n = 1$,

$$y(n = 1) = y(n-1) + x(n) = y(n = 0) + x(1) = 1 + 0 = 1.$$

Similarly, $y(n = 2) = 1$, and it can be shown that the output satisfies $y(n) = 1$ for $n \geq 0$. This second example shows that the system is BIBO-unstable, and yet for some bounded inputs the output will still be bounded. In fact, an energized student might be able to show another example featuring an unbounded input that offers a bounded output for this same system(!) The point is that lack of BIBO stability does *not* mean that bounded inputs will always yield bounded outputs, and unbounded inputs will always yield unbounded outputs. Lack of BIBO stability merely means that there exist some bounded inputs for which the output is unbounded.

Example 3: Some systems are unstable for a wider range of inputs. To see this, take $\alpha = 2$, meaning that $y(n) = 2y(n-1) + x(n)$. Let us revisit the impulse input of Example 2, $x(n) = \delta(n)$. Earlier, for $\alpha = 1$ we saw that the output is bounded. Now we have $y(n) = 0$ for $n < 0$, $y(n = 0) = 1$, and

$$y(n = 1) = 2y(n-1) + x(n) = 2y(n = 0) + x(1) = 2 \cdot 1 + 0 = 2.$$

Similarly, $y(2) = 4$, and in general it can be shown for positive n that $y(n) = 2^{n+1}$. We have seen that for the same input, $x(n) = \delta(n)$, increasing the feedback constant α from 1 to 2 makes the system appear less stable. Moreover, it can be shown for any $\alpha > 1$ that the output to this impulse will be unbounded.

BIBO stability and the impulse response: The slides mention the property that a system H is BIBO system if and only if $\sum_k |h(k)| < \infty$. Recall that $h(n)$ is the impulse response, and we have actually computed the impulse response. For example, for $\alpha = 1$ we saw that $h(n) = y(n) = 0$ for $n < 0$, $h(n) = 1$ for $n = 0$, and for $n > 0$ the output continues being 1. This can be succinctly expressed as $h(n) = u(n)$. Because

$$\sum_k |h(k)| = \sum_{k=-\infty}^{+\infty} |u(k)| = \sum_{k=0}^{+\infty} |1| = \infty,$$

the system is unstable. Indeed, when the input was a step function, $x(n) = u(n)$, the example above showed that the output $y(n)$ became unbounded.

Let us now consider a smaller feedback constant, namely $\alpha = \frac{1}{2}$. Simliar to before, $h(n) = 0$ for $n < 0$, and for $n = 0$ we have that $h(n) = 1$. Let us now consider $n = 1$,

$$y(n = 1) = \frac{1}{2}y(n-1) + x(n) = \frac{1}{2}y(n = 0) + x(1) = \frac{1}{2} \cdot 1 + 0 = \frac{1}{2},$$

where we remind the reader that we are computing the impulse response, and so the input is $x(n) = \delta(n)$, whose value is zero at all nonzero values of n. Similarly, $h(n = 2) = \frac{1}{4}$, and in fact it can be shown that $h(n) = 0.5^n u(n)$.

What about BIBO stability? To determine whether the feedback system involving $\alpha = \frac{1}{2}$ is stable, we compute the sum,

$$\sum_k |h(k)| = \sum_{k=-\infty}^{+\infty} |0.5^k u(k)| = \sum_{k=0}^{+\infty} 0.5^k = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2.$$

Because the infinite summation, which can be denoted using the ℓ_1 norm, i.e., $\|h\|_1$, is bounded, this feedback system is indeed BIBO stable.