Signal Denoising

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Motivation: Consider some continuous time signal $x_a(t)$, which is corrupted by noise $n(t)$. That is, we observe noisy measurements $y(t) = x_a(t) + n(t)$, and our goal is to recover the original $x_a(t)$.

This task appears in many applications. For example, in audio processing you may have a record that was scratched, and you want to recover the original music signal, x_a, from the scratched record, $y = x_a + n$. Other applications include radar signal processing and image processing (in the latter case, the image is a two-dimensional signal).

Main idea: Although this task may seem daunting, we can approach it by exploiting various types of *structure* in the signal. Here are some types of structure that may be present.

- The original input signal $x_a(t)$ is band-limited, whereas the noise $n(t)$ is not. In this case, $x_a(t)$ might have a *sparse representation* in the Fourier domain, meaning that a few coefficients will be large whereas most Fourier coefficients are small. In contrast, the noise is presumably present in all coefficients, because it is unstructured. To see this idea in action, you can run the following Matlab code.

  ```
  N=1000; % length of signal
  x=cos(0.26*(1:N)); % input
  n=randn(1,N); % noise
  y=x+n; % noisy observations
  plot(1:100,x(1:100),1:100,y(1:100));
  pause % take a look at the plot
  xf=fft(x);
  yf=fft(y);
  plot(1:N,abs(xf),1:N,abs(yf));
  ```

  It can be seen that the Fourier spikes corresponding to the cosine (the signal x) tower over the noise in the Fourier domain.

- The original $x_a(t)$ and noise $n(t)$ occupy the same spectral band, but in some frequency segments it is known that x_a contains more energy than n, whereas in other segments n has more energy than x_a. The solution to this sort of setting is to use a filter that passes more of $y = x_a + n$ (multiply by a number close to 1) in frequencies where x_a dominates, and to pass less of y (multiply by a number close to 0) where the noise n dominates.

- Instead of Fourier coefficients, maybe the input signal has some other type of convenient structure. For example, $x_a(t)$ could be a *piecewise constant* signal that transitions between different values at times $\ldots < T_{-1} < T_0 < T_1 < \ldots$, where $T_i \in \mathbb{R}$, and $x_a(t)$ takes the constant value V_i for $T_i \leq t < T_{i+1}$. If the noise n is unstructured as before, a denoising algorithm may be able to recover or denoise x from the noisy observations y. A possible way to do so is to search for an explanation $\widehat{x}$ for a possible input that matches the measurements well.

Image denoising: Our Matlab project involves image denoising. In images, different types of mechanisms may corrupt the original image. For example, if an image was acquired by some sensor, then random fluctuations in current may create small noise of a Gaussian-like nature. In other types of sensors, there may occasionally be a faulty pixel, which will create a large noise; this latter setting is often called salt and pepper noise.

You will see while working on the project that Gaussian noise can be handled by a linear filter. In contrast, for salt and pepper noise, a non-linear median filter can be quite useful. Indeed, part of the objectives of the Matlab project are to show you that non-linear signal processing can be useful (and potentially interesting to some of you).