

Solutions for Active Learning Questions

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Problem 1.1: Classify the signals below as: one/multi dimensional; single/multi channel; continuous/discrete time; digital/analog amplitude.

Solution: The way how I approach this example is to explain that there are different ways to interpret some of these. First, for closing prices of stocks, they are in discrete time, because the prices are evaluated at the close of each trading day. But prices could be interpreted as discrete (for example, prices could be in units of US cents), in which case the signal is digital; but if a stock price could increase in increments of cents up to many thousands of dollars, then interpreting it as continuous valued would be reasonable. Similarly, if we evaluate the prices of multiple stocks, we would have a multi-channel signal, whereas for one stock it would be a single channel.

For color movies, “old fashioned” movie theaters projected from film, which could be interpreted as being an analog signal, because the spatial domain and amplitude of the signal are both continuous in this case. In contrast, modern movies are recorded digitally. In both “old” film and modern digital systems, there are typically several dozen video frames per second, meaning that the signal has a discrete characteristic in the time domain (again, the spatial dimensions could be interpreted in different ways for traditional or modern movies). And due to the color nature of the movie (not to mention the extra audio), the signal is multi-channel. Finally, with two spatial dimensions and a third time dimension, the movie is multi-dimensional (then again, the audio might have different characteristics in terms of its dimensionality).

For weight and height measurements of a child every month, this is multi-channel, because we are measuring both the weight and the height. The signal is evaluated in discrete time, because it’s sampled monthly. The amplitudes could be continuous valued (if measured using analog scales and so on) or digital.

Example 1.4.4: Consider $x_a(t) = 3 \cos(2000\pi t) + 5 \sin(6000\pi t) + 10 \cos(12000\pi t)$. What is the Nyquist rate? If we sample at $F_s = 5000$ Hz, what is the discrete time signal? Finally, what analog signal is obtained using ideal sinc interpolation?

Solution: The components of the signal have frequencies $F_1 = 1000$, $F_2 = 3000$, and $F_3 = 6000$ cycles per second (Hz). The largest of these is F_3 , implying that the Nyquist rate is $2F_3 = 12000$ samples per second.

If we sample at $F_s = 5000$ Hz, then the discrete time signal is

$$\begin{aligned}
 x(n) &= x_a(t)|_{t=n/F_s} \\
 &= 3 \cos(2000\pi(n/5000)) + 5 \sin(6000\pi(n/5000)) + 10 \cos(12000\pi(n/5000)) \\
 &= 3 \cos(0.4\pi n) + 5 \sin(1.2\pi n) + 10 \cos(2.4\pi n) \\
 &= 3 \cos(0.4\pi n) + 5 \sin((1.2 - 2)\pi n) + 10 \cos((2.4 - 2)\pi n) \\
 &= 3 \cos(0.4\pi n) + 5 \sin(-0.8\pi n) + 10 \cos(0.4\pi n) \\
 &= 13 \cos(0.4\pi n) - 5 \sin(0.8\pi n).
 \end{aligned}$$

Because the second and third components (corresponding to frequencies $F_2 = 3000$ and $F_3 = 6000$) should have been sampled at more than 6000 and 12000 samples per second, respectively, while the sampling rate F_s was lower, both components will be aliased.

Finally, regarding sinc interpolation, the first component is reconstructed perfectly. In contrast, due to aliasing the other components are not. The third component is mapped to $10 \cos(2000\pi t)$, because it is aliased to the same digital frequency as the first component, i.e., $f_1 = 0.2$ and $f_3 = 0.2$. The second component is aliased to $f_2 = 0.4$, and ideal interpolation (n becomes $F_s t$) yields $-5 \sin(4000\pi t)$. In summary, the output of ideal interpolation is $13 \cos(2000\pi t) - 5 \sin(4000\pi t)$.

Correlation example: Consider $x(n)$ such that $x(n=0) = 2$, $x(n=1) = 1$, else $x(n) = 0$ for other time indices. We will compute the autocorrelation sequence $r_{xx}(l)$ for $l = -1, 0, +1$, and other values of l .

Solution: Autocorrelation is defined as $r_{xx}(l) = \sum_{n=-\infty}^{+\infty} x(n)x(n-l)$. For $l = -1$,

$$r_{xx}(l = -1) = \sum_{n=-\infty}^{+\infty} x(n)x(n+1).$$

The product within the summation is nonzero only when $n = 0$, in which case $x(n)x(n+1) = x(0)x(1) = 2 \cdot 1 = 2$.

For $l = 0$,

$$r_{xx}(l = 0) = \sum_{n=-\infty}^{+\infty} x(n)x(n) = \sum_n (x(n))^2.$$

We sum over the squares of $x(0)$ and $x(1)$, yielding $r_{xx}(0) = 2^2 + 1^2 = 5$.

For $l = +1$,

$$r_{xx}(l = +1) = \sum_{n=-\infty}^{+\infty} x(n)x(n-1).$$

The product within the summation is nonzero only when $n = 1$, in which case $x(n)x(n-1) = x(1)x(0) = 1 \cdot 2 = 2$.

Finally, for other values of l , the products of the form $x(n)x(n-l)$ are always zero, and these autocorrelations are thus also zero.

Example of z -transform: What is the z -transform and region of convergence (ROC) of the following signals: $x_1(n) = \{1, 2, 5\}$ and $x_2(n) = \{1, \underline{2}, 5\}$?

Solution: For $x_1(n)$,

$$\begin{aligned}X_1(z) &= \sum_{n=-\infty}^{+\infty} x_1(n)z^{-n} \\&= x_1(n=0)z^{-0} + x_1(n=1)z^{-1} + x_1(n=2)z^{-2} \\&= 1z^0 + 2z^{-1} + 5z^{-2} \\&= 1 + 2z^{-1} + 5z^{-2}.\end{aligned}$$

For the second signal,

$$\begin{aligned}X_2(z) &= \sum_{n=-\infty}^{+\infty} x_2(n)z^{-n} \\&= x_2(n=-1)z^{-(-1)} + x_2(n=0)z^{-0} + x_2(n=1)z^{-1} \\&= 1z^1 + 2z^0 + 5z^{-1} \\&= z + 2 + 5z^{-1}.\end{aligned}$$

You thought that was it, right? No, we still need to compute the ROCs for the two signals. For $x_1(n)$, when $z = 0$ we can't compute z^{-1} and z^{-2} in $X_1(z)$, and so $ROC_1 = \{z : z \neq 0\}$. For $x_2(n)$, the z -transform is again undefined for $z = 0$ due to having $5z^{-1}$ in the transform. But we also have z in the transform, meaning that $X_2(z)$ is undefined for $z = \infty$. (Note that in the complex domain we interpret infinity to mean any infinitely large value of z , which could be interpreted as a circle of infinitely large radius.) In summary, $ROC_2 = \{z : z \notin \{0, \infty\}\}$, because the transform is undefined for $z = 0$ and $z = \infty$.

Example of one-sided z -transform: What is the one-sided z -transform and region of convergence (ROC) of the following signals: $x_1(n) = \{\underline{1}, 2, 5\}$ and $x_2(n) = \{1, \underline{2}, 5\}$?

Solution: The key point in this example is that it is quite similar to the active learning example for regular (two-sided) z -transforms. For $x_1(n)$, it is nonzero only for non-negative time indices. Therefore, $X_1^+(z) = X_1(z) = 1 + 2z^{-1} + 5z^{-2}$, and $ROC_1 = \{z : z \neq 0\}$.

For $x_2(n)$, we must re-compute the one-sided z -transform,

$$\begin{aligned}X_2^+(z) &= \sum_{n=0}^{+\infty} x_2(n)z^{-n} \\&= x_2(n=0)z^{-0} + x_2(n=1)z^{-1} \\&= 2z^0 + 5z^{-1} \\&= 2 + 5z^{-1}.\end{aligned}$$

For the ROC, the one-sided transform no longer contains the z term of the double-sided transform, and so the only location in the complex plane where the transform is undefined is $z = 0$, i.e., $ROC_2 = \{z : z \neq 0\}$.

Difference equation: Consider the difference equation, $y(n) = 0.5y(n-1) + 2$ with initial condition $y(-1) = 0$. Compute $y(n)$ for $n \in \{0, 1, 2, 3\}$. Can you see a pattern for the values of $y(n)$?

Solution: We will compute these one by one.

$$\begin{aligned}
 y(n=0) &= 0.5 & y(n-1=-1)+ & 2 = & 0.5 \cdot 0 + 2 = & 2, \\
 y(n=1) &= 0.5 & y(n-1=0)+ & 2 = & 0.5 \cdot 2 + 2 = & 3, \\
 y(n=2) &= 0.5 & y(n-1=1)+ & 2 = & 0.5 \cdot 3 + 2 = & 3.5, \\
 y(n=3) &= 0.5 & y(n-1=2)+ & 2 = & 0.5 \cdot 3.5 + 2 = & 3.75.
 \end{aligned}$$

It can be seen that the steady state for large n will be $\lim_{n \rightarrow \infty} x(n) = 4$, because $0.5 \cdot 4 + 2 = 2 + 2 = 4$. That said, we seem to be converging toward the steady state value exponentially, and this gap shrinks from $4 - y(n=0) = 4 - 2 = 2$ for $n=0$ to $4 - y(n=1) = 4 - 3 = 1$ for $n=1$, $4 - y(n=2) = 4 - 3.5 = 0.5$ for $n=2$, and $4 - y(n=3) = 4 - 3.75 = 0.25$ for $n=3$. A function that yields this behavior is $y(n) = (4 - 2 \cdot 0.5^n)u(n)$.

Example 5.1.2: Consider the moving average filter,

$$y(n) = \frac{1}{3} [x(n-1) + x(n) + x(n+1)].$$

This filter corresponds to taking the input $x(n)$ and convolving it with $h = \{1/3, 1/3, 1/3\}$. That is, the moving average filter is linear time invariant (LTI), and its impulse response is h . We will compute frequency response.

Solution: What is $H(\omega)$? We take the discrete time Fourier transform of the impulse response h ,

$$H(\omega) = \frac{1}{3} [e^{-j\omega(-1)} + e^{-j\omega(0)} + e^{-j\omega(1)}] = \frac{1}{3} [e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3} [1 + 2 \cos(\omega)].$$

What is $|H(\omega)|$? Note that $H(\omega)$ is real valued. For ω such that $H(\omega) \geq 0$, the phase is zero, and $|H(\omega)| = H(\omega) = \frac{1}{3} [1 + 2 \cos(\omega)]$. For other ω , we have $H(\omega) < 0$, the phase is π (leading to $H(\omega)$ being negative), and $|H(\omega)| = -H(\omega) = -\frac{1}{3} [1 + 2 \cos(\omega)]$.

What is $\Theta(\omega)$? Recall that $\Theta(\omega)$ is the phase. We have seen that it is zero or π based on the sign of $H(\omega)$.

Example 5.1.5: Consider an LTI system whose impulse response is $h(n) = (\frac{1}{4})^n u(n)$. Our input will be $x(n) = (\frac{1}{2})^n u(n)$, and we want to determine the spectrum (Fourier transform) of the output, $y(n)$.

Solution: Whats the frequency response, $H(\omega)$? We will compute the Fourier transform of

the impulse response,

$$\begin{aligned}
 H(\omega) &= \sum_{n=-\infty}^{+\infty} h(n)e^{-j\omega n} \\
 &= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{4}\right)^n u(n)e^{-j\omega n} \\
 &= \sum_{n=0}^{+\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} \\
 &= \sum_{n=0}^{+\infty} \left(\frac{1}{4}e^{-j\omega}\right)^n \\
 &= \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.
 \end{aligned}$$

Because the structure of the signal $x(n)$ resembles that of $h(n)$, a similar derivation yields that its Fourier transform satisfies

$$X(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Our goal in this question was to compute the spectrum of the output, $y(n)$. Recall that $Y(\omega)$ is a product of $X(\omega)$ and $H(\omega)$, because in the time domain $y(n)$ is the convolution of $x(n)$ and $h(n)$. Therefore,

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

This form for $Y(\omega)$ can be simplified, for example using a partial fraction expansion.

Example 5.4.1: Consider a system with a double pole,

$$H(z) = \frac{b_0}{(1 - pz^{-1})^2}.$$

We want to compute the parameters b_0 and p such that $H(\omega = 0) = 1$ and $|H(\omega = \pi/4)|^2 = \frac{1}{2}$.

Solution: We begin by converting the requirement that $H(\omega = 0) = 1$ to the z -transfer domain. That is, $\omega^* = 0$ corresponds to $z^* = e^{j\omega^*} = e^0 = 1$.

What does $H(z^* = 1) = 1$ mean in terms of the parameters b_0 and p ? The constraint means that

$$H(z^* = 1) = \frac{b_0}{(1 - p1^{-1})^2} = \frac{b_0}{(1 - p)^2} = 1.$$

That is $b_0 = (1 - p)^2$.

Recall the second requirement, $|H(\omega = \pi/4)|^2 = \frac{1}{2}$. This means that $z^* = e^{j\pi/4}$. Plugging this into the transfer function,

$$H(z = e^{j\pi/4}) = \frac{b_0}{(1 - pe^{-j\pi/4})^2}.$$

Aliasing example: Consider a signal $x_a(t)$ whose Fourier transform has a triangular profile, with $X(F)$ rising from zero at $F = -1$ to 1 at $F = 0$, and then declining back to zero at $F = +1$ (the Fourier transform is zero outside the range $F \in [-1, +1]$). We are asked to sketch the Fourier transform of $x_a(t)\Delta_T(t)$ for three value of T .

Solution: Because $x_a(t)$ fills the bandwidth from $F = -1$ to $F = +1$, the Nyquist rate is $F_s = 2$. Next, we consider a sampling period $T = 1/2$, which means that we are sampling at the Nyquist rate. The triangular Fourier characteristic will be copied to the left and right, and each copy will be shifted by $1/T = 2$ units in the frequency domain. Because the triangle's width is 2, the copies of the triangle will touch each other. Moreover, the amplitude will change, because the amplitudes of the deltas in the frequency domain representation of $\Delta_{T=1/2}(t)$ are $1/T = 2$. That is, the amplitude of the shifted triangles will double.

The next part involves sampling with period $T = 1/5$, which corresponds to rate 5, which exceeds the Nyquist rate. The shifted triangles of width 2 will now be spaced 5 units apart in the frequency domain, meaning that they no longer touch. Additionally, the magnitude is now multiplied by a factor of $1/T = 5$.

The last part of the question involves sampling with period $T = 1$, which corresponds to rate 1. This rate is lower than the Nyquist rate, and we will get aliasing. The shifted triangles of width 2 will now be spaced one unit apart in the frequency domain, meaning that they will overlap significantly. In fact, it can be seen that due to one triangle increasing in amplitude while another decreases, overall the Fourier response will be flat. Because $1/T = 1$, the amplitudes will peak at 1, as in the original triangle. (Note, however, that these will no longer be peaks. Rather, the flat Fourier response will *always* be 1, i.e., $\mathcal{F}\{x_a(t)\Delta_{T=1}(t)\} = 1$, which corresponds to a single delta centered at the origin in the time domain, i.e., $x_a(t)\Delta_{T=1}(t) = \delta(t)$.)

DFT example 7.1.2: Consider a finite duration signal $x(n)$ whose values are 1 when $0 \leq n \leq L - 1$ and 0 when $L \leq n \leq N - 1$. We will compute the Fourier transform, $X(\omega)$.

Solution: The computation of the transform proceeds as follows,

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} \tag{1}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \tag{2}$$

$$= \sum_{n=0}^{L-1} 1 \cdot e^{-j\omega n} \tag{3}$$

$$= \frac{1 - e^{-jL\omega}}{1 - e^{-j\omega}}, \tag{4}$$

where (1) uses the definition for the Fourier transform, (2) exploits the finite duration of $x(n)$, (3) uses the structure of the signal, and (4) involves sums of geometric series. This final form can be simplified as follows,

$$\begin{aligned} X(\omega) &= \frac{1 - e^{-jL\omega}}{1 - e^{-j\omega}} \\ &= \frac{[e^{+jL\omega/2} - e^{-jL\omega/2}] e^{-jL\omega/2}}{[e^{+j\omega/2} - e^{-j\omega/2}] e^{-j\omega/2}} \end{aligned} \quad (5)$$

$$\begin{aligned} &= \frac{\frac{[e^{+jL\omega/2} - e^{-jL\omega/2}]}{2j} e^{-jL\omega/2}}{\frac{[e^{+j\omega/2} - e^{-j\omega/2}]}{2j} e^{-j\omega/2}} \end{aligned} \quad (6)$$

$$= \frac{\sin(L\omega/2)}{\sin(\omega/2)} e^{-j(L-1)\omega/2}. \quad (7)$$

where (5) is algebra for exponents, (6) divides both the numerator and denominator by $2j$, which leads to expressions for sine functions in (7).

Computing DFT using linear algebra: Consider the signal $x = [0 \ 1 \ 2 \ 3]^T$, where \cdot^T denotes the transpose operator. We will compute the DFT of x in several steps.

Solution: The first step is to compute w_4 , which is the root of one where the signal is of length $N = 4$,

$$w_4 = e^{-j2\pi/N} = e^{-j2\pi/4} = e^{-j\pi/2}.$$

It can be seen that $w_4 = -j$.

The next step involves computing the matrix W_4 ,

$$\begin{aligned} W_4 &= \begin{bmatrix} w_4^{0\cdot0} & w_4^{0\cdot1} & w_4^{0\cdot2} & w_4^{0\cdot3} \\ w_4^{1\cdot0} & w_4^{1\cdot1} & w_4^{1\cdot2} & w_4^{1\cdot3} \\ w_4^{2\cdot0} & w_4^{2\cdot1} & w_4^{2\cdot2} & w_4^{2\cdot3} \\ w_4^{3\cdot0} & w_4^{3\cdot1} & w_4^{3\cdot2} & w_4^{3\cdot3} \end{bmatrix} \\ &= \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}, \end{aligned}$$

where the last line substitutes $w_4 = -j$.

We now compute the DFT vector, X_4 , by multiplying the signal x_4 by W_4 ,

$$X_4 = W_4 x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 + 2j \\ -2 \\ -2 - 2j \end{bmatrix}.$$

This result can be verified using Matlab's fft command.

Linear phase: Consider the symmetric $h_1(n) = \{+1, \underline{0}, +1\}$. We will derive the frequency response, $H_1(\omega)$ and why it has linear phase. After that, the same will be derived for the anti-symmetric $h_2(n) = \{-1, \underline{0}, +1\}$.

Solution: The derivation for $H_1(\omega)$ follows,

$$\begin{aligned} H_1(\omega) &= \sum_{n=-\infty}^{+\infty} h_1(n)e^{-j\omega n} \\ &= h_1(n = -1)e^{-j\omega(-1)} + h_1(n = 0)e^{-j\omega(0)}h_1(n = +1)e^{-j\omega(+1)} \\ &= 1 \cdot e^{+j\omega} + 0 \cdot e^0 + 1 \cdot e^{-j\omega} \\ &= e^{+j\omega} + e^{-j\omega} \\ &= 2 \cos(\omega). \end{aligned}$$

This expression, $H_1(\omega)$, is real-valued. Therefore, its phase is always 0 or π , which is linear phase. (In other words, the phase $\Theta(\omega)$ has zero slope with possible discontinuities between 0 and π when the sign of the cosine flips.)

Next, we consider the anti-symmetric sequence,

$$H_2(\omega) = -e^{+j\omega} + e^{-j\omega} = (-2j) \frac{e^{+j\omega} - e^{-j\omega}}{2j} = -2j \sin(\omega).$$

The phase is now purely imaginary, meaning that it is $\pm\pi/2$. Again, the phase is linear with zero slope.