

# Poles, Causality, and Stability

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**LTI systems in z domain:** (Section 3.5 in book.) Consider a linear and time invariant (LTI) system with rational transfer function  $H(z) = \frac{B(z)}{A(z)}$ , where  $B(z)$  contains the zeros of the transfer function and  $A(z)$  contains poles. Similarly, suppose that the input signal  $x(n)$  can also be expressed in a rational fashion in the z domain,  $X(z) = \frac{N(z)}{Q(z)}$ . How about the output  $y(n)$  of our system? In the z domain,  $Y(z)$  is the product of our two rational z transforms,

$$Y(z) = X(z) \cdot H(z) = \frac{B(z)N(z)}{A(z)Q(z)}.$$

We will soon see that the poles of this expression have a pivotal role in appreciating the structure (in both the z and time domains) of the output signal.

To see this, we define some notation. Suppose that the poles of  $H$  are denoted by  $p_1, p_2, \dots, p_N$ , and that the poles in the expression for  $X(z)$  appear at locations  $q_1, \dots, q_L$ . We further assume that none of the zeros coincide with any of the poles (if they did indeed coincide, the zeros and poles would cancel out, which would require some refinements in our analysis). It can be shown that the z transform of the output,  $Y(z)$ , can be expressed as a sum of first order pole terms,

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{B_k}{1 - q_k z^{-1}},$$

where the first summation contains first order terms relating to poles of the LTI system  $H$ , and the second summation contains first order terms relating to  $X(z)$ . Taking the inverse z transform, it can be seen that in the time domain the output signal  $y(n)$  obeys

$$y(n) = \sum_{k=1}^N A_k (p_k)^n u(n) + \sum_{k=1}^L B_k (q_k)^n u(n),$$

where the causal  $u(n)$  terms could be replaced by anti-causal terms of the form  $u(-n - 1)$  depending on the ROC, as we have seen before. The first summation is often called the *natural response*, it is the output of the system to its initial conditions. The second summation is the *forced response*, and it depends on the input.

We want the natural response to be *transient*, which means that we want it to decay to zero. To do so, we need the poles of  $H(z)$  to be inside the unit circle, i.e.,  $|p_k| < 1$ , in which case  $\lim_{n \rightarrow \infty} A_k (p_k)^n = 0$ . We will soon relate this requirement to stability.

**Pole location and time response:** To keep things simple, suppose that a pole  $p$  corresponds to a causal signal. Suppose further that  $p \in \mathbb{R}$ , it is a real-valued pole. (For real-valued signals and systems, complex-valued poles and zeros will always appear in complex conjugate pairs, which can be addressed separately.)

- $|p| < 1$  and  $p > 0$  – The response will decay to zero, because  $|p| < 1$ ; the signs of the response will all be the same.
- $|p| < 1$  and  $p < 0$  – The response will decay to zero; the signs will flip every sample due to the negative sign in  $p$ .
- $|p| = 1$  and  $p > 0$  – In this case  $p = +1$ , and the response will maintain the same value, i.e., it will have the form  $h(n) = \text{const} \cdot u(n)$ .
- $|p| = 1$  and  $p < 0$  – In this case  $p = -1$ , and the signs will flip every sample, i.e.,  $h(n) = \text{const} \cdot (-1)^n u(n)$ .
- $p > 1$  - Irrespective of the sign,  $h(n)$  will blow up. This is a bad impulse response.

And a small comment about double poles. So far we have discussed 5 cases of positions for real-valued single poles. When we have a double pole, the response takes on the form  $n\alpha^n u(n)$ , which will blow up even for  $|\alpha| = 1$ . Moreover, multiplication by  $n$  will increase the magnitude of  $h(n)$  unless the pole is comfortably inside the unit circle. Therefore, in order to maintain a margin of error we prefer to only use poles that are comfortably inside the unit circle. This rule of thumb also allows to deal with additional glitches such as the precision of arithmetic. (Imprecise arithmetic may cause a pole inside yet near the unit circle to create problems like an unstable pole, which is outside the unit circle.)

**BIBO stability:** We have mentioned stability a few times so far. What is it? Consider a system  $H$  with impulse response  $h(n)$ . The system is said to be stable in the bounded input bounded output (BIBO) sense if for every bounded input  $x$ , i.e.,  $\max_n |x(n)| < \infty$ , the output is also bounded, i.e.,  $\max_n |y(n)| < \infty$ . (Note that these maxima can be written conveniently using an  $\ell_\infty$  norm as follows:  $\|x\|_\infty = \max_n |x_n|$ . Consequently, BIBO stability happens when  $\|x\|_\infty < \infty$  implies that  $\|y\|_\infty < \infty$ .)

It can be shown that a system  $H$  is BIBO stable if and only if its impulse response has finite  $\ell_1$  norm, i.e.,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty. \quad (1)$$

Let us now evaluate the z transform on the unit circle, i.e.,  $|z| = 1$ ,

$$\begin{aligned} |H(z)| &= \left| \sum_{n=-\infty}^{+\infty} h(n)z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{+\infty} |h(n)z^{-n}| \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{+\infty} |h(n)||z|^{-n} \\
&= \sum_{n=-\infty}^{+\infty} |h(n)|,
\end{aligned}$$

where the inequality ( $\leq$ ) is due to the triangle inequality, and we employ the fact that  $|z|^{-n} = 1^{-n} = 1$ . Note, however, that for a BIBO stable system the last line in this derivation will be finite (1), and we conclude that

$$|H(z)| < \infty, \forall |z| = 1.$$

That is, the ROC of a BIBO stable system contains the unit circle.

We have seen that causal systems have ROCs outside some circle, and so the ROC of a BIBO causal system has the form  $ROC = \{z : |z| > r\}$ , where  $0 < r < 1$ . Similarly, anti-causal systems (nonzero impulse response,  $h(n) \neq 0$ , only for negative time,  $n < 0$ ) have ROCs inside some circle, and so the ROC of a BIBO anti-causal system has the form  $ROC = \{z : |z| < r\}$ , where  $r > 1$ .

**Example 1.** *Let us discuss a comprehensive example that should bring together pole location, causality, and stability. Consider Example 3.5.2, where*

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}.$$

*In Part (a), the question asks what the ROC must be in order for the system  $H$  to be BIBO stable; Part (b) asks whether the system is stable if we know that it is causal. Instead of answering these two parts directly, let us consider the two components of the  $z$  transform, and for each component consider its causal and anti-causal versions.*

*The first component is  $\frac{1}{1 - \frac{1}{2}z^{-1}}$ , and the second component is  $\frac{2}{1 - 3z^{-1}}$ . If the first component is causal, then  $h(n)$  contains an expression of the form  $(\frac{1}{2})^n u(n)$ , and we denote this case by 1C (C for causal); else the first component is anti-causal,  $h(n)$  contains an expression of the form  $-(\frac{1}{2})^n u(-n - 1)$ , and we denote this case by 1A (A for anti-causal). Similarly, for the second component we have the causal  $2 \cdot 3^n u(n)$  and anti-causal  $-2 \cdot 3^n u(-n - 1)$  denoted by 2C and 2A, respectively.*

*We can now discuss the four combinations.*

- *1C and 2C – Both components are causal, and so in this case the system  $H$  is causal. However,  $3^n u(n)$  blows up, and the system is unstable. We can see this by considering the ROC,*

$$ROC_{1C,2C} = ROC_{1C} \cap ROC_{2C} = \{z : |z| > \frac{1}{2}\} \cap \{z : |z| > 3\} = \{z : |z| > 3\},$$

*and this ROC does not contain the unit circle.*

- *1C and 2A* – The first component is causal and the second component is anti-causal, and so  $H$  is neither causal nor anti-causal. Interestingly, both  $(\frac{1}{2})^n u(n)$  and  $-3^n u(-n-1)$  converge to zero, which implies that the system  $H$  is BIBO stable. The stability can be observed using the ROC,

$$ROC_{1C,2A} = ROC_{1C} \cap ROC_{2A} = \{z : |z| > \frac{1}{2}\} \cap \{z : |z| < 3\} = \{z : \frac{1}{2} < |z| < 3\},$$

which contains the unit circle.

- *1A and 2C* – The first component is anti-causal and the second component is causal, and so  $H$  is neither causal nor anti-causal. Because the 1A component  $-(\frac{1}{2})^n u(-n-1)$  blows up, as does the 2C component  $3^n u(n)$ , the system is unstable. Looking at the ROC,

$$ROC_{1A,2C} = ROC_{1A} \cap ROC_{2C} = \{z : |z| < \frac{1}{2}\} \cap \{z : |z| > 3\} = \{\},$$

which is the empty set. The ROC is empty precisely because  $h(n)$  blows up whether moving toward the positive time direction ( $n \rightarrow +\infty$ ) or toward the negative time direction ( $n \rightarrow -\infty$ ).

- *1A and 2A* – Both components are anti-causal, and so  $H$  is anti-causal. Because the 1A component  $-(\frac{1}{2})^n u(-n-1)$  blows up, the system is unstable. Looking at the ROC,

$$ROC_{1A,2A} = ROC_{1A} \cap ROC_{2A} = \{z : |z| < \frac{1}{2}\} \cap \{z : |z| < 3\} = \{z : |z| < \frac{1}{2}\},$$

which does not contain the unit circle.