4.23.2014 ①

Problems related to Chapter 7

7.5 Compute $\sum_{n=0}^{N-1} x_1(n)\, x_2(n)$.

(a) $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$, $0 \le n \le N-1$.

$x_2(n) = x_1(n)$.

Parseval gives us $\sum_{n=0}^{N-1} (x_1(n))^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X_1(k)|^2$.

want to compute $X_1(k)$.

Recall that $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$

$$= \frac{1}{2} e^{+j\frac{2\pi n}{N}} + \frac{1}{2} e^{-j\frac{2\pi n}{N}}.$$

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn}$$

$$= \left\{\frac{1}{2}\sum_{n=0}^{N-1} e^{+j\frac{2\pi n}{N}} e^{-j\frac{2\pi}{N}kn}\right\} + \left\{\frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}} e^{-j\frac{2\pi}{N}kn}\right\}$$

$$= \frac{1}{2}\underbrace{\left\{\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(1-K)}\right\}}_{\begin{array}{ll}\frac{N}{2} & K=0 \\ 0 & \text{else}\end{array}} + \frac{1}{2}\underbrace{\left\{\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(-1-K)}\right\}}_{\begin{array}{ll}\frac{N}{2} & K=-1\ (K=N-1) \\ 0 & \text{else}\end{array}}$$

In summary, $X(K) = \begin{cases} \frac{1}{2}N & K = 1, N-1 \\ 0 & \text{else} \end{cases}$

$$\sum_{n=0}^{N-1} |x_1(n)|^2 = \frac{1}{N}\sum_{K=0}^{N-1} |X_1(k)|^2$$

$$= \frac{1}{N}\left\{\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2\right\}$$

$$= \frac{1}{N} \cdot \frac{N^2}{4} \cdot 2$$

$$= \frac{N}{2}.$$

we verified this numerically with matlab.

(b) $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$, $x_2(n) = \sin\left(\frac{2\pi n}{N}\right)$

we have computed $X_1(k) = \begin{cases} \frac{N}{2} & k = 1, N-1 \\ 0 & \text{else} \end{cases}$

To compute $X_2(k)$, observe that

$$x_2(n) = \frac{1}{2j} e^{+j\frac{2\pi}{N}n} - \frac{1}{2j} e^{-j\frac{2\pi}{N}n}.$$

Therefore,

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j\frac{2\pi}{N}nk}$$

$$= \frac{1}{2j}\left\{\sum_{n=0}^{N-1} e^{+j\frac{2\pi}{N}n} \cdot e^{-j\frac{2\pi}{N}nk}\right\} - \frac{1}{2j}\left\{\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}} e^{-j\frac{2\pi nk}{N}}\right\}$$

$$= \begin{cases} \frac{N}{2j} & k = 1 \\ -\frac{N}{2j} & k = N-1 \\ 0 & \text{else} \end{cases}$$

Finally, Parseval gives us

$$\sum_{n=0}^{N-1} x_1(n)\, x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2^*(n)$$

x_2 is real-valued

Parseval

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k)^*$$

$$= \frac{1}{N}\left\{\frac{N}{2} \cdot \left(\frac{1}{2j}\right)^* N + \frac{N}{2} \cdot \left(-\frac{1}{2j}\right)^* N\right\}$$

$$= N \cdot \frac{1}{4}\left\{\left(\frac{1}{j}\right)^* - \left(\frac{1}{j}\right)^*\right\}$$

$$= 0.$$

Again, this was verified numerically.

7.7 If $X(k)$ is the DFT of $x(n)$, determine the DFT of

$$x_c(n) = x(n) \cdot \cos\left(\frac{2\pi k_0 n}{N}\right).$$

Approach #1 $x_c(n) = \frac{1}{2} x(n) e^{+j\frac{2\pi k_0 n}{N}} + \frac{1}{2} x(n) e^{-j\frac{2\pi k_0 n}{N}}$

$$\Rightarrow X_c(k) = \frac{1}{2} X(k-k_0) + \frac{1}{2} X(k+k_0).$$

Approach #2 $X_c(k) = \sum_{n=0}^{N-1} x_c(n) \cdot e^{-j\frac{2\pi}{N}kn}$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{2} x(n) e^{+j\frac{2\pi k_0 n}{N}} + \frac{1}{2} x(n) e^{-j\frac{2\pi k_0 n}{N}} \right\} e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{2} \underbrace{\left\{ \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}n(k-k_0)} \right\}}_{X(k-k_0)} + \frac{1}{2} \underbrace{\left\{ \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}n(k+k_0)} \right\}}_{X(k+k_0)}$$

$$= \frac{1}{2} X(k-k_0) + \frac{1}{2} X(k+k_0).$$

we used the DFT definition,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}.$$

7.4 For $x_1(n) = \cos(\frac{2\pi n}{N})$ and $x_2(n) = \sin(\frac{2\pi n}{N})$,

(a) want to compute $x_1(n) \circledast x_2(n)$.

$$x_1(n) \circledast x_2(n) = IDFT\{DFT\{x_1(n)\} \cdot DFT\{x_2(n)\}\}.$$

Recall,

$$X_1(k) = \begin{cases} \frac{N}{2} & k=1, N-1 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad X_2(k) = \begin{cases} \frac{N}{2j} & k=1 \\ -\frac{N}{2j} & k=N-1 \\ 0 & \text{else} \end{cases}$$

$$X_1(k) \cdot X_2(k) = \begin{cases} \frac{N^2}{4j} & k=1 \\ -\frac{N^2}{4j} & k=N-1 \\ 0 & \text{else} \end{cases} = \frac{N}{2} \cdot X_2(k).$$

Therefore,

$$\begin{aligned} x_1(n) \circledast x_2(n) &= IDFT\{\tfrac{N}{2} X_2(k)\} \\ &= \tfrac{N}{2}\, IDFT\{X_2(k)\} \\ &= \tfrac{N}{2}\, x_2(n) \\ &= \tfrac{N}{2} \cdot \sin(\tfrac{2\pi n}{N}) \end{aligned}$$

we verified this numerically using matlab.

(c) $x_1(n) \circledast x_1(n) = IDFT\{DFT\{x_1(n)\}^2\}$

$$= IDFT\begin{cases} \frac{N^2}{4} & k=1, N-1 \\ 0 & \text{else} \end{cases}$$

$$= \frac{N}{2} IDFT\{X_1(k)\}.$$

$\Rightarrow x_1(n) \circledast x_1(n) = \frac{N}{2} x_1(n)$.

(d) Similarly, $x_2(n) \circledast x_2(n) = IDFT\left\{\begin{matrix} (\frac{N}{2j})^2 & k=1 \\ (-\frac{N}{2j})^2 & k=N-1 \\ 0 & \text{else} \end{matrix}\right\}$

$$= -\frac{N}{2} IDFT\{X_1(k)\} = -\frac{N}{2} \cdot x_1(n).$$

7.16 An LTI system has impulse response,

$$h(n) = \delta(n) - \tfrac{1}{4}\delta(n-k_0).$$

An engineer computes the N-point DFT, $H(k)$, $N = 4k_0$. The engineer defines

$$g(n) = \mathrm{IDFT}\{G(k)\},$$

$$G(k) = \frac{1}{H(k)}.$$

We start by computing $H(k)$,

$$H(k) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \underbrace{1 \cdot e^{-j\frac{2\pi}{N}k\cdot 0}}_{=1} - \frac{1}{4}\underbrace{e^{-j\frac{2\pi}{N}k\cdot k_0}}_{e^{-j\frac{2\pi k k_0}{4k_0}} = e^{-j\frac{\pi k}{2}} = (-j)^k}$$

$$= 1 - \tfrac{1}{4}(-j)^k.$$

$$G(k) = \frac{1}{H(k)}$$

$$= \frac{1}{1 - \frac{1}{4}(-j)^k}.$$

Because $G(k) \cdot H(k) = 1$, and $\mathrm{IDFT}\{1\} = \delta(n)$, the circular convolution between g and h is the impulse response,

$$g \circledast h = \delta(n).$$

However, the ordinary convolution $g * h$ is different, and g is <u>not</u> the inverse system.