Homework 03 – Problem 3

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Let me describe Problem 3 in Homework 03 based on the parameters given to me by the software. My signal was

$$x(n) = \begin{cases} (\frac{2}{9})^n & n \geq 0 \\ (\frac{1}{4})^{-n} & n < 0 \end{cases},$$

and the filter was

$$h(n) = \begin{cases} (\frac{5}{8})^n & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

The question asked to compute the convolution, $y(n) = x(n) * h(n)$, and the corresponding region of convergence.

To solve this question, we compute the z transforms of x and z. First, the z transform of $(\frac{2}{9})^n u(n)$ is $\frac{1}{1-\frac{2}{9}z^{-1}}$ where the ROC is $\{z : |z| > \frac{2}{9}\}$. Second, the z transform of $(\frac{1}{4})^{-n}u(-n-1)$, which can be rewritten as $4^n u(-n-1)$, is $-\frac{1}{1-4z^{-1}}$ where the ROC is $\{z : |z| < 4\}$. Combining these two z transforms, and due to linearity of the z transform ($Z(x_1) + Z(x_2) = Z(x_1 + x_2)$ where $Z(\cdot)$ denotes z transformation and x_1, x_2 are two signals),

$$X(z) = \frac{1}{1 - \frac{2}{9}z^{-1}} - \frac{1}{1 - 4z^{-1}},$$

and the ROC is $\{z : \frac{2}{9} < |z| < 4\}$. The z transform of the filter h obeys

$$H(z) = \frac{1}{1 - \frac{5}{8}z^{-1}},$$

and the ROC is $\{z : \frac{5}{8} < |z|\}$.

Having computed the z transforms (including ROCs) of the two signals x and h, we can compute the z transform of the convolution by multiplying the two z transforms. That is,

$$Y(z) = X(z) \cdot H(z) = \left(\frac{1}{1 - \frac{2}{9}z^{-1}} - \frac{1}{1 - 4z^{-1}} \right) \frac{1}{1 - \frac{5}{8}z^{-1}},$$

and the ROC is the intersection of the two previous ROCs, which is $\{z : \frac{5}{8} < |z| < 4\}$.

To compute $y(n)$, we must take the inverse z transform of $Y(z)$, which contains products of pairs of terms of the form $\frac{1}{1+\alpha z^{-1}}$ and thus seems quite involved. Fortunately, a partial fraction expansion can considerably simplify the remainder of the solution. We proceed as follows:

$$\begin{aligned} Y(z) &= \frac{\left(1-4z^{-1}\right)-\left(1-\frac{2}{9}z^{-1}\right)}{\left(1-\frac{2}{9}z^{-1}\right)\left(1-4z^{-1}\right)\left(1-\frac{5}{8}z^{-1}\right)} \\ &= \frac{a}{1-\frac{2}{9}z^{-1}}+\frac{b}{1-4z^{-1}}+\frac{c}{1-\frac{5}{8}z^{-1}}, \end{aligned}$$

where a, b, and c are constants, which will soon be computed. We compute the first numerator,

$$\left(1-4z^{-1}\right)-\left(1-\frac{2}{9}z^{-1}\right)=-\frac{34}{9}z^{-1},$$

and the second numerator,

$$\begin{aligned} & a\left(1-4z^{-1}\right)\left(1-\frac{5}{8}z^{-1}\right)+b\left(1-\frac{2}{9}z^{-1}\right)\left(1-\frac{5}{8}z^{-1}\right)+c\left(1-\frac{2}{9}z^{-1}\right)\left(1-4z^{-1}\right) \\ = \; & a\left(1-\frac{37}{8}z^{-1}+\frac{5}{2}z^{-2}\right)+b\left(1-\frac{61}{72}z^{-1}+\frac{5}{36}z^{-2}\right)+c\left(1-\frac{38}{9}z^{-1}+\frac{8}{9}z^{-2}\right) \\ = \; & 1\cdot(a+b+c)+z^{-1}\cdot\left(-\frac{37}{8}a-\frac{61}{72}b-\frac{38}{9}c\right)+z^{-2}\cdot\left(\frac{5}{2}a+\frac{5}{36}b+\frac{8}{9}c\right), \end{aligned}$$

where we note in passing that $\frac{2}{9}+\frac{5}{8}=\frac{2\cdot 8+5\cdot 9}{8\cdot 9}=\frac{16+45}{72}=\frac{61}{72}$. We can solve for the three constants a, b, and c by solving the following linear algebra problem,

$$\begin{bmatrix} 1 & 1 & 1 \\ -\frac{37}{8} & -\frac{61}{72} & -\frac{38}{9} \\ \frac{5}{2} & \frac{5}{36} & \frac{8}{9} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ -\frac{34}{9} \\ 0 \end{bmatrix}.$$

In Matlab, this is solved as following,

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A=[1 1 1; -37/8 -61/72 -38/9; 5/2 5/36 8/9];
b=[0; -34/9;0];
inv(A)*b % solution
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yielding the solution $a=-0.5517$, $b=-1.1852$, and $c=1.7369$. With these constants, we can compute the inverse z transform as follows,

$$y(n)=-0.5517(\frac{2}{9})^n u(n)+1.1852(4)^n u(-n-1)+1.7369(\frac{5}{8})^n u(n),$$

where we emphasize that we had to flip the sign of b in the inverse z transform, because the z transform of $\frac{b}{1-4z^{-1}}$ is $-b4^n u(-n-1)$.