ECE 421 Introduction to Signal Processing

Final Exam – Spring 2013

May 8, 2013

Question 1

(a) Compute the circular convolution of the sequences $x_1(n) = [2, 3, 4, 2]$ and $x_2(n) = [1, 2, 1, 0]$. (In all sequences throughout this test, the first location corresponds to $n = 0$ or $k = 0$.)

Solution:

$$
\begin{aligned}
y(0) &= \sum([2\ 3\ 4\ 2]. * [1\ 0\ 1\ 2]) = 2*1+3*0+4*1+2*2 = 10, \\
y(1) &= \sum([2\ 3\ 4\ 2]. * [2\ 1\ 0\ 1]) = 4+3+2 = 9, \\
y(2) &= \sum([2\ 3\ 4\ 2]. * [1\ 2\ 1\ 0) = 2+6+4 = 12, \\
y(3) &= \sum([2\ 3\ 4\ 2]. * [0\ 1\ 2\ 1] = 3+8+2 = 13.
\end{aligned}
$$

(b) Suppose that $x1$ and $x2$ have both been entered into Matlab. Please provide Matlab code that calculates the circular convolution using the DFT and IDFT.

Solution: ifft(fft(x1).*fft(x2)).

Question 2

(a) Compute the four-point ($N = 4$) DFT $X(k)$ of the sequence $x(n) = [2, 3, 4, 2]$.

Solution:

$$
\begin{aligned}
X(k) &= W_4 x \\
&= \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2+3+4+2 \\ 2-3j-4+2j \\ 2-3+4-2 \\ 2+3j-4-2j \end{bmatrix} \\
&= \begin{bmatrix} 11 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}.
\end{aligned}
$$

(b) Compute the inverse DFT $x(n)$ of $X(k) = [7, j, -3, -j]$.
Solution: The solution closely resembles the previous part,

$$\begin{aligned}
x(n) &= \frac{1}{N}(W_4)^{-1}X \\
&= \frac{1}{4}\begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^{-1} & w_4^{-2} & w_4^{-3} \\ w_4^0 & w_4^{-2} & w_4^{-4} & w_4^{-6} \\ w_4^0 & w_4^{-3} & w_4^{-6} & w_4^{-9} \end{bmatrix}\begin{bmatrix} 7 \\ j \\ -3 \\ -j \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 7 \\ j \\ -3 \\ -j \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 7 + j - 3 - j \\ 7 + j^2 + 3 + j^2 \\ 7 - j - 3 + j \\ 7 - j^2 + 3 - j^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}.
\end{aligned}$$

Question 3

(a) Compute the Four-point DFT of the sequence $x(n) = [1, 1, 2, 2]$ using a radix-2 FFT algorithm.
Solution: First, arrange $[1, 1, 2, 2]$ as a matrix,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Next, compute DFT's of length 2 of rows,

$$G = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix},$$

where DFT-2 involves the matrix

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Next, modulate the second row of G using an approach resembling the following Matlab command,

```
G(2,:)=G(2,:).* exp(-j*2*pi*(0:N/2-1)/N);
```

This calculation yields

$$G = \begin{bmatrix} 3 & -1 \\ 3 & j \end{bmatrix}.$$

Taking DFT-2 over columns,

$$X = \begin{bmatrix} 6 & 0 \\ -1+j & -1-j \end{bmatrix}.$$

Finally, reshaping X as a column yields $X(k) = [6, -1+j, 0, -1-j]$.
(b) Suppose that N_1 is some prime number and that $N = (N_1)^2$. Approximately how many arithmetic operations are needed to

compute the DFT using a divide and conquer approach?

Solution: We will need to (*i*) compute N_1 DFT's of length N_1; (*ii*) multiply all $(N_1)^2 = N$ terms by modulation terms; and (*iii*) compute N_1 additional DFT's of length N_1. To keep things simple, suppose that each DFT of length N_1 requires roughly $c_1(N_1)^2 = c_1 N$ operations, and that the modulation of N terms requires $c_2 N$ operations. The total will be $2N_1$ DFT's totaling $2N_1 \cdot c_1 N = 2c_1 N^{1.5}$ operations and $c_2 N$ for modulations. The grand total of $2c_1 N^{1.5} + c_2 N$ will be dominated by the $N^{1.5}$ term for large N.

Question 4

Consider an analog filter with system function

$$H_a(s) = \frac{s+1}{(s+1)^2 + 16}.$$

(a) Convert the analog filter into a digital IIR filter using a bilinear transformation, where the digital filter is supposed to have a resonant frequency of $\omega_r = \pi/3$.

Solution: This question follows Handout #25. The analog filter has resonance frequency $\Omega_r = 4$, which is the square root of the 16 in the denominator. We map that analog frequency to the digital $\omega_r = \pi/3$ where the transformation obeys $\omega = 2\tan^{-1}(\Omega T/2)$. We want

$$\frac{\pi}{3} = 2\tan^{-1}\left(\frac{\Omega_r T}{2}\right),$$

which implies

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{4T}{2} = 2T.$$

Therefore, $T = \frac{1}{2\sqrt{3}}$. Next, $s = \frac{2(z-1)}{T(z+1)} = \frac{4\sqrt{3}(z-1)}{(z+1)}$. Substituting into $H_a(s)$,

$$\begin{aligned} H(z) &= H_a\left(s = \frac{4\sqrt{3}(z-1)}{(z+1)}\right) \\ &= \frac{\frac{4\sqrt{3}(z-1)}{(z+1)} + 1}{\left(\frac{4\sqrt{3}(z-1)}{(z+1)} + 1\right)^2 + 16}. \end{aligned}$$

In order to simplify the calculation, in the rest of question 4 please use a digital IIR filter with tranfer function

$$H(z) = \frac{10z^2 + 2z - 10}{100z^2 - 200z + 100}$$

instead of the answer you got in part (a).

(b) Where are the zero(s) and pole(s) of $H(z)$?
solution: The denominator is

$$100z^2 - 200z + 100 = 100(z^2 - 2z + 1) = 100(z-1)^2,$$

and there is a double pole at $p_1 = p_2 = 1$. The numerator is

$$10z^2 + 2z - 10,$$

and there are zeros at

$$z_{1,2} = -\frac{2}{2 \cdot 10} \pm \frac{\sqrt{2^2 - 4 \cdot 10 \cdot (-10)}}{2 \cdot 10} = -0.1 \pm \frac{\sqrt{404}}{20} = -0.1 \pm \sqrt{1.01},$$

which can be shown to be $z_1 = -0.905$ and $z_2 = +1.105$.
(c) Sketch the pole-zero plot.
Solution: The sketch should include: (*i*) the real and imaginary axes of the z-plane; (*ii*) axes should be labeled; (*iii*) a double pole at $p_1 = p_2 = 1$; (*iv*) two zeros at $z_1 = -0.905$ and $z_2 = +1.105$. Note that one of the zeros will be close to the double poles.
(d) Suppose that this system is stable. What is the region of convergence (ROC) of this filter? Is $H(z)$ a causal system?
Solution: Stability means that the ROC includes the unit circle. There was a slight mistake with this question, and because the poles were on the unit circle it is impossible for the system to be stable. (I graded this part of the final leniently, of course.)