ECE 421 Introduction to Signal Processing

Final – Spring 2014

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Question 1

Let $h(n) = \{\underline{1}, 0, -1\}$ be the impulse response of an FIR filter, and let $x(n) = \{\underline{1}, 2, 1, 2\}$ be the input sequence. (The underline corresponds to time zero.)

(a) How many zeros need to be padded in $x(n)$ and $h(n)$, respectively, in order to avoid aliasing in the output, $y(n) = x * h(n)$, if we want to use the DFT and IDFT to calculate $y(n)$?

(b) Suppose that the zero padded versions of x and h have been defined in Matlab, and are called *xpad* and *hpad*, respectively. Please provide Matlab code that calculates $y(n)$ via DFT and IDFT.

(c) Denote the length of $x(n)$ after zero padding by L. Suppose that you got $L = 4$ in part **(a)**. Please compute the L point DFT of $x(n)$.

Question 2

If $x(n)$ is a periodic sequence with a period N, i.e.,

$$x(n) = x(n+N),$$

then $x(n)$ is also periodic with period $2N$. Let $X_N(k)$ denote the discrete Fourier transform (DFT) coefficients when $x(n)$ is considered to be periodic with period N, and let $X_{2N}(k)$ be the DFT coefficients when the period is assumed to be $2N$.

(a) Write the expression for $X_{2N}(k)$ using $x(n)$.

(b) Express the DFT coefficients $X_{2N}(k)$ in terms of $X_N(k)$. (Hints: because $x(n) = x(n+N)$, you can partition the sum into two parts. Note also that $1 + e^{-j\pi k}$ is either 0 or 2 depending on whether k is odd or even.)

Question 3

The first five points of the eight-point DFT of a real-valued sequence are $\{1, 1-j, 2, 2+j, -1\}$. Determine the remaining three points.

Question 4

Consider the signal $x_1(n) = \sin\left(\frac{2\pi n}{N}\right)$. Compute $\sum_{n=0}^{N-1}(x_1(n))^2$ using the DFT.

Question 5

A discrete time signal $x(n)$ has the form

$$x(n) = A_1 \cos(\omega_1 n) + A_2 \cos(\omega_2 n) + z(n),$$

where $z(n)$ is noise, and the frequencies ω_1 and ω_2 are unknown. You are given $N = 10^4$ samples of $x(n)$ for $n = 0, \ldots, N-1$.

(a) Suppose that the amplitudes of the cosines are identical, i.e., $A_1 = A_2$, and that these amplitudes are larger than the amplitude of the noise $z(n)$. What type of window seems well matched for this spectral estimation problem? Justify your answer by describing how you would estimate ω_1 and ω_2.

(b) How would your answer change if you are told that $A_1 = A$, $A_2 = 30A$, and that $|\omega_1 - \omega_2| > 0.2\pi$? (As before, A is larger than the amplitude of the noise $z(n)$.) Justify your answer.

Question 6

Suppose that we have two integers x_a and x_b that can be expressed using N digits in base-D,

$$x_a = \sum_{n=0}^{N-1} \widetilde{x_a}(n) D^n, \tag{1}$$

$$x_b = \sum_{n=0}^{N-1} \widetilde{x_b}(n) D^n. \tag{2}$$

That is, x_a and x_b are the actual numbers and $\widetilde{x_a}$ and $\widetilde{x_b}$ are corresponding signals in base D. (For example, we could have $x_a = 12$, $x_b = 13$, $\widehat{x_a} = \{\underline{2}, 1\}$, and $\widehat{x_b} = \{\underline{3}, 1\}$.)

Our goal in this question is to derive a fast algorithm to multiply x_a and x_b.

(a) The classical approach to multiplication, which most of us learned in elementary school, proceeds as follows,

$$\begin{aligned} x_c &= x_a \cdot x_b \\ &= \left(\sum_{n=0}^{N-1} \widetilde{x_a}(n) D^n \right) \left(\sum_{n=0}^{N-1} \widetilde{x_b}(n) D^n \right) \\ &= \sum_{n_a, n_b = 0}^{N-1} \widetilde{x_a}(n_a) \widetilde{x_b}(n_b) D^{n_a + n_b}. \end{aligned}$$

Please show that $x_c = \sum_{n=0}^{2N-2} D^n \cdot \{\widetilde{x_a} * \widetilde{x_b}\}(n)$. That is, convolution between $\widetilde{x_a}$ and $\widetilde{x_b}$ forms the entries of a signal $\widetilde{x_c}$ that corresponds to the product x_c.

(b) We can see that $x_c = x_a \cdot x_b$ can be computed using the following algorithm:

1. Form the sequences $\widetilde{x}_a$ and $\widetilde{x}_b$.
2. Compute the convolution $\widetilde{x}_c = \widetilde{x}_a * \widetilde{x}_b$.
3. Multiply each $x_c(n)$ by D^n.
4. Compute the sum of the product terms.

Show using this algorithm that the product of $x_a = 26$ and $x_b = 23$ is $x_c = 598$ (we use the decimal base $D = 10$). (In our example from the previous page, $\widehat{x_a} * \widehat{x_b} = \{\underline{6}, 5, 1\}$, and $6 \cdot 10^0 + 5 \cdot 10^1 + 1 \cdot 10^2 = 156$, which is the product of $x_a = 12$ and $x_b = 13$.)

(c) Suppose that x_a and x_b that can be expressed using N digits, and suppose further that the FFT of an input of length N can be computed using approximately $3N\log(N)$ floating point operations. Approximately how many operations would you need to compute x_c using the FFT to perform the convolution in the algorithm above?