ECE 421 Introduction to Signal Processing

Final Exam – Spring 2019

April 29, 2019

Question 1 (DFT properties)
A real-valued discrete time signal, $x(n)$, is of length $N = 6$. We are given several values of its discrete Fourier transform (DFT):

$$X(0) = 10, \quad X(1) = 1 + 2j, \quad X(2) = 2 - j, \quad X(3) = 2.$$

(a) Determine the missing values of $X(k)$ for $k \in \{4, 5\}$.

Solution: Because the signal is real-valued, DFT coeffiicents are conjugate symmetric: $X(4) = X(2)^* = 2 + j$ and $X(5) = X(1)^* = 1 - 2j$.

(b) We are given another length-6 discrete time signal, $y(n)$, whose DFT, $Y(k)$, is related to $X(k)$ as follows,

$$Y(k) = X(k)[1 + e^{-j\pi k}].$$

Compute $Y(k)$ for $k \in \{0, 1, 2, 3, 4, 5\}$. (If you are not sure about your answers in part (a), you may assume that $X(4) = 6 - 5j$ and $X(5) = -7 + 2j$.)

Solution: Note that $1 + e^{-j\pi k} = 1 + (-1)^k$, which is 2 for even k and 0 for odd k. Therefore, for odd k we have $Y(1) = Y(3) = Y(5) = 0$, and for even k $Y(0) = 2X(0) = 2(10) = 20$, $Y(2) = 2X(2) = 2(2 - j) = 4 - 2j$, and $Y(4) = 2X(4) = 2(2 + j) = 4 + 2j$.

(c) Consider the DFT time shift property, $x(n - m)_N \leftrightarrow X(k)e^{-j2\pi km/N}$, where $x(n-m)_N$ denotes time indices that are cyclical with respect to modulo N. Use the time shift property to express $y(n)$ in terms of $x(n)$. (We expect a response such as $y(n) = 2x(n) - 17$ or $y(n) = x(n+1)_N - x(n-3)_N$.)

Solution: Because $N = 6$ and $e^{-j\pi k} = e^{-j\pi \frac{3k}{N}}$, the shift $m = 3$. Owing to the time shift property and linearity, $y(n) = x(n) + x(n-m)_N = x(n) + x(n-3)_6$.

Question 2 (short questions)
In the questions below, make sure to justify your answers.
(a) True or false: Could a linear time invariant (LTI) discrete time system with the following transfer function,

$$H(z) = \frac{(z-1)(z+1)}{(z-2)^2(z+\frac{1}{2})},$$

be both causal and bounded input bounded output (BIBO)?

Solution: This transfer function contains poles at $p_{1,2} = 2$ and $p_3 = -\frac{1}{2}$. For the LTI system to be causal and BIBO, all poles must be inside the unit circle. However, p_1 and p_2 are outside the unit circle, meaning that this LTI system cannot be both causal and BIBO.

(b) True or false: Could a finite impulse response (FIR) filter whose impulse response is anti-symmetric have a low pass frequency response?

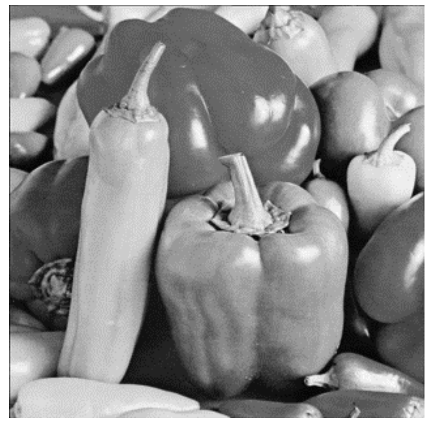
Solution: An anti-symmetric filter has a magnitude frequency response comprised of a superposition of sine functions. However, sine functions are zero at $\omega = 0$, meaning that the response of an anti-symmetric FIR is zero, and it cannot be low pass.

(c) We have seen how to use circular convolution to compute linear convolution. Suppose now that two discrete time input signals, $x_a(n)$ and $x_b(n)$, are both of length N. Can you compute their circular convolution, i.e., $x_a \circledast x_b$ of length N, using linear convolution? (If you cannot, explain why. If you can, explain how.)

Solution: Circular convolution can be first implemented by computing linear convolution, resulting in a sequence x_c of length $2N-1$ (coefficient indices will range from $n = 0$ to $n = 2N-2$), and then "wrapping around" coefficients N through $2N-2$. For example, consider $x_a = \{\underline{1}, 2\}$ and $x_b = \{\underline{3}, 2\}$. The linear convolution is $x_c = 3\{\underline{1}, 2\} + 2\{\underline{0}, 1, 2\} = \{\underline{3}, 8, 4\}$. There is one element at index $N = 2N-2 = 2$, $x_c(2) = 4$, which is wrapped around to index $n = 0$, meaning that the output is $x_d = \{\underline{7}, 8\}$. A quick Matlab evaluation using the cconv command results in the same output.

Question 3 (DCT-based image compression)
In Project 5, you developed a lossy image compression algorithm based on quantizing discrete cosine transform (DCT) coefficients. Consider the two images below, Image 1 and Image 2. They are both outputs of the DCT-based algorithm using different compression settings, and you will evaluate how these settings impact various aspects of the image compression system. Make sure to justify your answers.

(a) Image 1



(b) Image 2

(a) Which image requires a higher coding rate?

Solution: The higher coding rate is required for the higher quality Image 2.

(b) Which image uses a larger step size in the quantizer?

Solution: A larger step size implies the lower quality Image 1.

(c) Which image corresponds to a larger percentage of nonzero quantized DCT coefficients?

Solution: A larger percentage of nonzero quantized coefficients corresponds to a smaller step size, and thus the higher quality Image 2.

Question 4 (linear convolution using DFT)

Let $h(n) = \{\underline{1}, 0, 1\}$ be the impulse response of an FIR filter, and let $x(n) = \{\underline{1}, 1\}$ be the input sequence. (Underlines correspond to time index zero.) In this question, you will compute the linear convolution, $y = h * x$, both directly and using the DFT and inverse DFT (IDFT).

(a) We want to compute the linear convolution, $y = h*x$. What is the length of y? Over what range of time indices is $y(n)$ defined?

Solution: The length is $len(y) = len(h) + len(x) - 1 = 3 + 2 - 1 = 4$. The indices range from 0 to $len(y) - 1 = 3$.

(b) Compute y using the standard definition for convolution,

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k).$$

(If you are unsure about your answers in part (a), you can compute $y(n)$ for indices $n \in \{-1, 0, 1, 2\}$.)

Solution: The linear convolution is $\{\underline{1}, 0, 1\} * \{\underline{1}, 1\} = \{\underline{1}, 0, 1\} + \{\underline{0}, 1, 0, 1\} = \{\underline{1}, 1, 1, 1\}$.

(c) To compute y using the DFT and IDFT, we begin by zero padding h and x to form two modified sequences, $\widetilde{h}$ and $\widetilde{x}$, whose lengths are identical. What are $\widetilde{h}$ and $\widetilde{x}$?

Solution: We zero pad to length 4, $\widetilde{h} = \{\underline{1}, 0, 1, 0\}$ and $\widetilde{x} = \{\underline{1}, 1, 0, 0\}$.

(d) Compute $\widetilde{H}(k)$ and $\widetilde{X}(k)$, the DFTs of the zero padded sequences $\widetilde{h}$ and $\widetilde{x}$, respectively. We use the following definition for the DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}.$$

(If you are unsure about your answers in part (c), you can assume that $\widetilde{h}(n) = \{\underline{0}, 1, 0, 1\}$ and $\widetilde{x}(n) = \{\underline{0}, 2, 2, 0\}$.)

Solution: The DFT can be computed by following the equation; another option is to use the matrix form. Either way, the correct values are $\widetilde{H}(k) = \{\underline{2}, 0, 2, 0\}$ and $\widetilde{X}(k) = \{\underline{2}, 1-j, 0, 1+j\}$. The "you can assume" solutions are $\widetilde{H}(k) = \{\underline{2}, 0, -2, 0\}$ and $\widetilde{X}(k) = \{\underline{4}, -2-2j, 0, -2+2j\}$.

(e) Recall that $Y(k)$, the DFT of the linear convolution, $y = h * x$, is the product of the DFTs of $\widetilde{h}$ and $\widetilde{x}$. That is, $Y(k) = \widetilde{H}(k)\widetilde{X}(k)$. Compute $Y(k)$. (If you are unsure about your answers in part (d), you can assume that $\widetilde{H}(k) = \{\underline{3}, 2+j, 0, 2-j\}$ and $\widetilde{X}(k) = \{\underline{1}, 1, 1, 1\}$.)

Solution: Multiplying $\widetilde{H}(k) = \{\underline{2}, 0, 2, 0\}$ and $\widetilde{X}(k) = \{\underline{2}, 1-j, 0, 1+j\}$ yields $Y(k) = \{\underline{4}, 0, 0, 0\}$. (Note that we designed h and x such that the DFT's of their zero padded versions contain zeros, and thus $Y(k)$ is relatively easy to calculate.) For the "you can assume" part, multiplying $\widetilde{H}(k) = \{\underline{3}, 2+j, 0, 2-j\}$ and $\widetilde{X}(k) = \{\underline{1}, 1, 1, 1\}$ yields $Y(k) = \{\underline{3}, 2+j, 0, 2-j\}$. Again, we designed this to be simple by having all ones in our suggested version of $\widetilde{X}(k)$.

(f) Compute y by applying the IDFT to $Y(k)$. We use the following definition for the IDFT,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{+j2\pi kn/N}.$$

(If you worked through the entire question correctly, then your answer should be identical to that in part (b); congratulations! And if you are unsure about your answer in part (e), you can assume that $Y(k) = \{\underline{1}, 0, 1, 0\}$.)

Solution: Because $Y(k) = \{\underline{4}, 0, 0, 0\}$ has a delta form, we expect $y(n)$ to be a constant vector. Indeed, computing the IDFT yields $y(n) = \{\underline{1}, 1, 1, 1\}$, which is identical to our answer in part (b). For the "you can assume" part, $y(n) = \{\underline{\tfrac{1}{2}}, 0, \tfrac{1}{2}, 0\}$.