ECE 421 Introduction to Signal Processing

Midterm 1 – Spring 2019

February 13, 2019

Question 1 (sampling)
This question considers continuous time analog signals being sampled at rate $F_s = 2000$ samples per second.

(a) What is the Nyquist frequency F_N, which limits the highest analog frequency of signals that can be sampled and later perfectly reconstructed with this sampling rate F_s?

Solution: The Nyquist rate is half the sampling rate,

$$F_N = F_s/2 = 1000$$

cycles per second. Note, however, that it might not be possible to perfectly sample and reconstruct signals at F_N itself; we can only guarantee this for frequencies below F_N.

(b) Find three different analog signals, $x_1(t)$, $x_2(t)$, and $x_3(t)$, that produce the discrete time signal

$$x(n) = \cos(0.2\pi n)$$

when sampled at F_s.

Solution: Consider $x(t) = \cos(2\pi F t)$, which when sampled will give

$$x(n) = x_a(t = n/F_s) = \cos(2\pi F n/F_s) = \cos(2\pi F/2000n).$$

We will determine three frequencies, F_1, F_2, and F_3. We want the sampled $x(n)$ to equal $\cos(0.2\pi n)$, and a natural choice for F_1 (the first solution among our three) satisfies

$$2\pi F_1/2000 = 0.2\pi,$$

implying that $F_1 = 200$ cycles per second. The more interesting solutions, F_2 and F_3, can involve

$$2\pi F_2/2000 = 2.2\pi, \quad 2\pi F_3/2000 = 4.2\pi,$$

implying that $F_2 = 2200$ cycles per second and $F_3 = 4200$ cycles per second. The three solutions are

$$x_1(t) = \cos(2\pi 200t), \quad x_2(t) = \cos(2\pi 2200t), \quad x_3(t) = \cos(2\pi 4200t).$$

(c) We are given the following continuous time analog signal,

$$x_a(t) = \sin(2\pi F_1 t) + \cos(2\pi F_2 t),$$

where $F_1 = 400$ Hz and $F_2 = 1700$ Hz. We sample $x_a(t)$ at a sampling rate $F_s = 2000$ Hz. What are the discrete time frequencies, f_1 and f_2, that correspond to F_1 and F_2? (Make sure that $f_1, f_2 \in [-0.5, +0.5)$ cycles per sample.)

Solution: We compute the discrete time signal $x(n)$,

$$\begin{aligned}
x(n) &= x_a(t = n/F_s) \\
&= \sin(2\pi F_1 n/F_s) + \cos(2\pi F_2 n/F_s) \\
&= \sin(0.4\pi n) + \cos(1.7\pi n) \\
&= \sin(0.4\pi n) + \cos((1.7 - 2)\pi n) \\
&= \sin(0.4\pi n) + \cos(-0.3\pi n) \\
&= \sin(0.4\pi n) + \cos(0.3\pi n) \\
&= \sin(2\pi 0.2 n) + \cos(2\pi 0.15 n) \\
&= \sin(2\pi f_1 n) + \cos(2\pi f_2 n).
\end{aligned}$$

The corresponding discrete time frequencies are $f_1 = 0.2$ and $f_2 = 0.15$ cycles per sample.

Question 2 (properties of signals)
Consider the following discrete time system T,

$$T[x(n)] = (x(n))^2.$$

Is T linear or non-linear? Time variant or time invariant? Causal? Make sure to justify your answer carefully.

Solution: The system T is clearly non-linear. To see this, take an input, $x_1(n)$. Multiply it by -1, i.e., $x_2(n) = -x_1(n)$. The outputs corresponding to both inputs are identical,

$$y_2(n) = T[x_2(n)] = (x_2(n))^2 = (-x_1(n))^2 = (x_1(n))^2 = T[x_1(n)] = y_1(n).$$

If T was linear, then $T[x_2(n)]$ should equal $-T[x_1(n)]$, which is not the case, and T is non-linear.
To evaluate time variance or invariance, note that $y(n)$ depends only on $x(n)$, and does not depend on other time indices. Therefore T is time invariant, and it is also causal.

Question 3 (LTI systems, stability, and z-transforms)
Consider a linear time invariant (LTI) discrete time system H whose impulse response h satisfies,

$$\begin{aligned} h(n) &= 0.5^{|n|} + \delta(n) \\ &= 0.5^n u(n) + 2^n u(-n-1) + \delta(n). \end{aligned}$$

(a) Determine the z-transform, $H(z)$.

Solution: The transform is computed as follows,

$$\begin{aligned} H(z) &= \mathcal{Z}\{0.5^n u(n)\} + \mathcal{Z}\{2^n u(-n-1)\} + \mathcal{Z}\{\delta(n)\} \\ &= \frac{1}{1-0.5z^{-1}} - \frac{1}{1-2z^{-1}} + 1 \\ &= \frac{z}{z-0.5} - \frac{z}{z-2} + 1. \end{aligned}$$

(b) Is H causal?

Solution: The component $0.5^n u(n)$ is causal while $2^n u(-n-1)$ is anti-causal. Therefore, overall H is not causal.

(c) What is the region of convergence (ROC), ROC_H?

Solution: Based on part (a), there are poles at $p_1 = 0.5$ and $p_2 = 2$. The first pole corresponds to a causal component and thus an ROC $\{|z| > 0.5\}$. The second pole is anti-causal, and corresponds to an ROC $\{|z| < 2\}$. Overall, ROC_H is the intersection of these two regions, and so $ROC_H = \{0.5 < |z| < 2\}$.

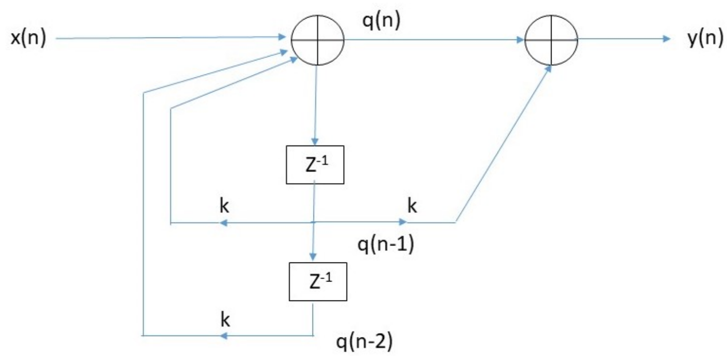
(d) Is H stable in the bounded input bounded output (BIBO) sense? Please answer in *two different ways.*

Solution: The first way to answer is to look at the sum of magnitudes of h,

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{-1} 2^n + 1 + \sum_{n=0}^{+\infty} 0.5^n = 1 + 1 + 2 = 4.$$

Because the summation is finite, H is BIBO stable. The second looks at the ROC. The unit circle is contained in the ROC, and so H is BIBO stable.

Question 4 (block diagrams, difference equations, and transfer functions)
Consider the block diagram below. A discrete time input signal, $x(n)$, is processed by a first block, resulting in $q(n)$, which is then processed by a second block, resulting in the discrete time output signal, $y(n)$. We also have a parameter, k, which controls the amplification of $q(n-1)$ and $q(n-2)$.



(a) Derive the transfer function, $H(z) = Y(z)/X(z)$. (Hint: it may be convenient for some of you to first derive a transfer function corresponding to the first block, $Q(z)/X(z)$, and then to the second block, $Y(z)/Q(z)$.

Solution: We see that

$$q(n) = x(n) + kq(n-1) + kq(n-2).$$

Taking the z-transform,

$$Q(z) = X(z) + kz^{-1}Q(z) + kz^{-2}Q(z).$$

Rearranging terms,

$$\frac{Q(z)}{X(z)} = \frac{1}{1 - kz^{-1} - kz^{-2}}.$$

Similarly,

$$y(n) = q(n) + kq(n-1).$$

Therefore,

$$Y(z) = Q(z)(1 + kz^{-1}) = X(z)\frac{1 + kz^{-1}}{1 - kz^{-1} - kz^{-2}},$$

and the transfer function is

$$H(z) = \frac{1 + kz^{-1}}{1 - kz^{-1} - kz^{-2}}.$$

(b) Suppose that the parameter k satisfies $k = -4$. Where are the poles of $H(z)$? (Hint: if you are not sure about your answer in part (a), you may assume that $H(z) = \frac{k+z^{-2}}{1+kz^{-2}}$.)

Solution: Using the true answer from part (a),

$$H(z) = \frac{1 + kz^{-1}}{1 - kz^{-1} - kz^{-2}} = \frac{z^2 - 4z}{z^2 + 4z + 4},$$

the poles are roots of the denominator, i.e., $z^2 + 4z + 4 = (z + 2)^2$. The solutions to this quadratic equation are both -2. That is, $p_1 = p_2 = -2$.
Using the hint,

$$H(z) = \frac{k + z^{-2}}{1 + kz^{-2}} = \frac{kz^2 + 1}{z^2 + k} = \frac{-4z^2 + 1}{z^2 - 4}.$$

The roots of the denominator are roots of $z^2 - 4 = (z + 2)(z - 2)$, and so $p_{1,2} = \pm 2$.

Question 5 (z-transform)
In this question, you will derive the z-transform of the ramp signal, $u_r(n) = nu(n)$. To do so, you will work through several parts.

(a) Consider a discrete time system whose input is the step function, $u(n)$, and whose output satisfies,

$$y(n) = y(n-1) + u(n),$$

where the output is initially at rest, meaning that $y(n) = 0$ for $n < 0$. Express $y(n)$ in the form,

$$y(n) = \alpha u_r(n) + \beta u(n) + \gamma \delta(n). \tag{1}$$

That is, compute the constants α, β, and γ. (Hint: compute $y(n)$ for several values of n.)

Solution: At time $n = 0$, $y(n = 0) = y(n = -1) + u(n = 0) = 0 + 1 = 1$. At time $n = 1$, $y(n = 1) = y(n = 0) + u(n = 1) = 1 + 1 = 2$. At time $n = 2$, $y(n = 2) = y(n = 1) + u(n = 2) = 2 + 1 = 3$. We can see that $y(n) = (n + 1)u(n) = u_r(n) + u(n)$. Therefore, $\alpha = 1$, $\beta = 1$, and $\gamma = 0$.

(b) Based on part (a), $Y(z) = \alpha U_r(z) + \beta \mathcal{Z}\{u(n)\} + \gamma \mathcal{Z}\{\delta(n)\}$. Compute the z-transforms of $\delta(n)$ and $u(n)$.

Solution: The z-transform of the delta function is

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{+\infty} \delta(n) z^{-n} = 1 \cdot z^{-0} = 1.$$

The z-transform of the step function is

$$U(z) = \mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{+\infty} u(n) z^{-n} = \sum_{n=0}^{+\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \ldots = \frac{1}{1 - z^{-1}}.$$

(c) Derive $Y(z)$, the z-transform of $y(n)$. To do so, recall that $y(n)$ is the output of the difference equation given in (1), and the z-transform of its input was already derived in part (b).

Solution: The z-transform of the difference equation is $Y(z) = z^{-1}Y(z) + U(z)$. Therefore,

$$Y(z) = \frac{U(z)}{1 - z^{-1}} = \frac{1}{(1 - z^{-1})^2}.$$

(d) You can now combine the above information to compute the z-transform of $u_r(n)$. Because there are many opportunities for glitches in this question, *this part will not be graded.*

Solution: The transform is

$$\begin{aligned} U_r(z) &= Y(z) - \beta \mathcal{Z}\{u(n)\} - \gamma \mathcal{Z}\{\delta(n)\} \\ &= \frac{1}{(1-z^{-1})^2} - 1 \cdot \frac{1}{1-z^{-1}} - 0 \cdot 1 \\ &= \frac{1-(1-z^{-1})}{(1-z^{-1})^2} \\ &= \frac{z^{-1}}{(1-z^{-1})^2}. \end{aligned}$$