Question 1 (sampling)
An analog signal, $x_a(t) = 4\cos(6\pi t)\cos(12\pi t)$, is sampled at a rate of $F_s = 10$ samples per unit time.

(a) Show that $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$. (Hint: you can use the identity, $\cos(\theta) = \frac{1}{2}[e^{+i\theta} + e^{-i\theta}]$.)

Solution: The derivation proceeds as follows,

$$\begin{aligned}
\cos(\alpha)\cos(\beta) &= \frac{1}{2}[e^{+i\alpha} + e^{-i\alpha}]\frac{1}{2}[e^{+i\beta} + e^{-i\beta}] \\
&= \frac{1}{4}[e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}] \\
&= \frac{1}{4}[e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}] + \frac{1}{4}[e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}] \\
&= \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)].
\end{aligned}$$

(b) Using Part (a), express $x_a(t)$ as a sum of two cosines.

Solution: Using Part (a),

$$\begin{aligned}
x_a(t) &= 4\cos(6\pi t)\cos(12\pi t) \\
&= 4 \times \frac{1}{2}[\cos(6\pi t + 12\pi t) + \cos(6\pi t - 12\pi t)] \\
&= 2[\cos(18\pi t) + \cos(6\pi t)].
\end{aligned}$$

(c) Compute the discrete time signal, $x(n)$, obtained by sampling $x_a(t)$ at rate $F_s = 10$ samples per unit time. Make sure to bring $x(n)$ to the form $c_1\cos(2\pi f_1 n) + c_2\cos(2\pi f_2 n)$, where c_1, c_2 are amplitudes, and f_1, f_2 are cycles per unit time. (Hint: if you are unsure about your answer in part (b), you may assume that $x_a(t) = 13\cos(2\pi t) - 7\cos(22\pi t)$.)

Solution: We sample the simplified form of $x_a(t)$ from Part (b), where $t = n/F_s$,

$$\begin{aligned}
x(n) &= x_a(t = n/10) \\
&= 2[\cos(18\pi n/10) + \cos(6\pi n/10) \\
&= 2[\cos((1.8-2)\pi n) + \cos(0.6\pi n)] \\
&= 2[\cos(0.2\pi n) + \cos(0.6\pi n).
\end{aligned}$$

The various constants are $c_1 = c_2 = 2$, $f_1 = 0.1$, and $f_2 = 0.3$.

The assuming option proceeds as follows,

$$\begin{aligned} x(n) &= x_a(t = n/10) \\ &= 13\cos(2\pi n/10) - 7\cos(22\pi n/10) \\ &= 13\cos(0.2\pi n) - 7\cos(0.2\pi n) \\ &= 6\cos(0.2\pi n). \end{aligned}$$

Owing to aliasing, we are left with a single cosine, $c = 6$, and $f = 0.1$.

(d) Was there aliasing in Part (c)? Make sure to justify your answer.

Solution: Yes, the $\cos(1.8\pi n)$ was aliased to $\cos(0.2\pi n)$.
The assuming option is similar.

Question 2 (properties of signals)
In each part below, we will specify a discrete time system and ask you about some of its properties. Make sure to justify your answers.

(a) Is the following time variant or time invariant?

$$y(n) = (x(n) + x(n-1))^3.$$

Solution: The power of 3, while nonlinear, does not affect time variance. This system is time invariant, because

$$y(n+k) = (x(n+k) + x(n+k-1))^3,$$

which is $y(n)$ for $x(n+k)$, an input x shifted by k time samples.

(b) Consider the following linear time invariant (LTI) discrete time system,

$$y(n) = y(n-1) + 2x(n).$$

Is the system stable in the bounded input bounded output (BIBO) sense?

Solution: Recall from class that the impulse response of the similar system, $y(n) = y(n-1) + x(n)$, is the step function, $u(n)$. Multiplying by 2 gives an impulse response, $h(n) = 2u(n)$, whose sum of magnitudes is infinite, and the system is not BIBO. Another way to see this is to evaluate the transfer function, $H(z) = \frac{2}{1-z^{-1}}$, and notice a pole on the unit circle.

(c) The following system is not LTI,

$$y(n) = \frac{1}{|x(n)|}.$$

Is it BIBO stable?

Solution: This one is not BIBO stable either. Similar to a question on one of our projects, you can take values $x(n)$ that approach zero, and $y(n)$ will increase without bound.

Question 3 (z-transform)
In this question, you will be analyzing several signals with similar structure.

(a) Compute the z-transform and specify the region of convergence (ROC) for the following discrete time signal,

$$x_a(n) = 0.5^n u(n) + 3^n u(n).$$

Solution: The z-transform is

$$X_a(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{1}{1 - 3z^{-1}},$$

where the first component has $ROC_{a1} = \{|z| > 0.5\}$, because it is causal, and the second component has $ROC_{a2} = \{|z| > 3\}$, because it is also causal. Taking the intersection,

$$ROC_a = ROC_{a1} \cap ROC_{a2} = \{|z| > 3\}.$$

(b) Compute the z-transform and specify the ROC for the following discrete time signal,

$$x_b(n) = 0.5^n u(n) - (3^n)u(-n-1).$$

(Note that the negative sign applies to the entire expression, $3^n u(n)$.)

Solution: The z-transform is

$$X_b(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{1}{1 - 3z^{-1}},$$

it is the same as before. However, the ROC is different, because the second component is anti-causal, and its ROC is $ROC_{b2} = \{|z| < 3\}$. Taking the intersection,

$$ROC_b = ROC_{b1} \cap ROC_{b2} = \{0.5 < |z| < 3\}.$$

(c) Explain why the following discrete time signal,

$$x_c(n) = -(0.5)^n u(-n-1) + 3^n u(n),$$

does not have a z-transform.

Solution: The ROCs of the two components are $ROC_{c1} = \{|z| < 0.5\}$ and $ROC_{c2} = \{|z| > 3\}$. These do not overlap, and so the ROC is empty,

$$ROC_c = ROC_{c1} \cap ROC_{c2} = \{\}.$$

Question 4 (difference equations)
Consider the following discrete time difference equation,

$$y(n) = 0.5y(n-1) + x(n) - 2x(n-1).$$

(a) Determine the transfer function, $H(z) = Y(z)/X(z)$, of the difference equation. It suffices to bring the transfer function to the form $H(z) = \frac{z-z_1}{z-p_1}$.

Solution: The transform of the difference equation is

$$Y(z) = 0.5Y(z)z^{-1} + X(z) - 2X(z)z^{-1}.$$

Rearranging terms,

$$Y(z)[1 - 0.5z^{-1}] = X(z)[1 - 2z^{-1}],$$

and the transfer function is

$$H(z) = \frac{1-2z^{-1}}{1-0.5z^{-1}} = \frac{z-2}{z-0.5}.$$

(b) Where are the zeros and poles of the transfer function? Sketch a corresponding pole-zero plot. (Hint: if you are unsure about your answer in Part (a), assume that $H(z) = \frac{z-2}{z-0.5}$.)

Solution: We have a zero at $z_1 = 2$ and pole at $p_1 = 0.5$.
The assuming options is the same.

(c) Perform a partial fraction expansion, and express $H(z)$ in the form $H(z) = A + \frac{B}{z-p_1}$.

Solution: The derivation proceeds as follows,

$$H(z) = \frac{z-2}{z-0.5} = A + \frac{B}{z-0.5} = \frac{A(z-0.5)+B}{z-0.5}.$$

We see that $A(z-0.5)+B = z-2$, and conclude that $A = 1$ and $B = -1.5$,

$$H(z) = 1 - \frac{1.5}{z-0.5}.$$

The assuming option is the same.
(d) Based on your answer in Part (c), compute the impulse response, $h(n)$, of the difference equation. You can assume that the difference equation is causal. (Hint: if you are unsure about your answer in Part (c), assume that $H(z) = 2 + \frac{3}{z-0.5}$.)

Solution: There was a glitch in this question, because the inverse transform of $\frac{1}{1-pz^{-1}} = \frac{z}{z-p}$ is $p^n u(n)$. Multiplying by z^{-1} (in the numerator, to cancel out the z) introduces a delay in the time domain. Owing to this glitch, I attempted to grade this part flexibly. Possible answers were $h(n) = \delta(n) - 1.5(0.5)^n u(n)$ for the transform from Part (c), and $h(n) = 2\delta(n) + 3(0.5)^n u(n)$ for the assuming option.

(e) Draw either a Direct Form I or Direct Form II implementation of the difference equation.

Solution: A simple approach is to partition the difference equation into two parts. In the first part,

$$v(n) = x(n) - 2x(n-1),$$

implemented using a single delay element that converts $x(n)$ to $x(n-1)$. In the second part,

$$y(n) = 0.5y(n-1) + v(n),$$

which again uses one delay element.

(f) We will now implement the difference equation in MATLAB. Most of the implementation is provided, and you only need to specify *one missing line* that computes the output y for time indices $n = 2, 3, \ldots, N$.

```
n = 1:N; % time index
x = ... % specify input x
y = x*0; % multiply by zero to initialize y
y(1) = x(1); % assume that y(n-1)=y(0)=0 and x(n-1)=x(0)=0
for n=2:N % fill out the rest of y
    y(n) = ... % <<---- COMPLETE THIS LINE BELOW
end
```

Solution: The missing line is y(n)=0.5*y(n-1)+x(n)-2*x(n-1)