Question 1 (Fourier transform)
Consider the following discrete time periodic signal,

$$x(n) = 3\cos\left(\frac{\pi n}{2}\right) + 4\cos\left(\frac{\pi n}{4}\right).$$

a) What is the period, $T_p$, of the signal? Make sure to justify your answer.

**Solution:** $x(n) = 3\cos\left(\frac{2\pi n}{4}\right) + 4\cos\left(\frac{2\pi n}{8}\right)$.
The smallest value that can be divided by 4 and 8 is 8. Thus $T_p = 8$.

b) Compute the Fourier transform of the signal.
**Solution:** $X[k] = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-\frac{2\pi jk}{8}n}$
$= \frac{1}{8}\sum_{n=0}^{7}\left[3\frac{e^{\frac{2\pi jn}{4}}+e^{\frac{-2\pi jn}{4}}}{2} + 4\frac{e^{\frac{2\pi jn}{8}}+e^{\frac{-2\pi jn}{8}}}{2}\right]e^{-\frac{2\pi jk}{8}n}$
$= \frac{1}{8}\sum_{n=0}^{7}\left[\frac{3}{2}\left(e^{\frac{2\pi j(-k+2)n}{8}} + e^{\frac{2\pi j(-k-2)n}{8}}\right) + 2\left(e^{\frac{2\pi j(-k+1)n}{8}} + e^{\frac{2\pi j(-k-1)n}{8}}\right)\right]$

Due to the orthogonality property of Fourier basis (sinusoidal functions):
For any $k \neq 2, 10, 18...$, $\frac{1}{8}\sum_{n=0}^{7} e^{\frac{2\pi j(-k+2)n}{8}} = 0$;
for any $k = 2, 10, 18...$, $\frac{1}{8}\sum_{n=0}^{7} e^{\frac{2\pi j(-k+2)n}{8}} = \sum_{n=0}^{7} e^{0} = 1$;
etc..
In summary:
$C_{2+8N} = C_{-2+8N} = \frac{3}{2}$;
$C_{-1+8N} = C_{1+8N} = 2$ for $N \in \mathbf{N}$;
otherwise,$C_k = 0$.

c) What is the average energy (or power), $P_x$, of the signal?

**Solution:** $P_x = (\frac{3}{2})^2 \cdot 2 + 2^2 \cdot 2 = \frac{25}{2}$.

Question 2 (transfer functions)
Consider a discrete time linear time invariant (LTI) system, $H$, whose impulse response is given by

$$h(n) = \left(\frac{1}{4}\right)^n u(n).$$

a) Compute the transfer function, $H(\omega)$, in the Fourier domain.

**Solution:** $H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$.
$H(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$.

b) Derive the magnitude and phase response of the system $H$.

**Solution:** $|H(\omega) = |\frac{1}{1-0.25cos\omega+0.25jsin\omega}| = \sqrt{\frac{1}{1.0625-0.5cos\omega}}$.
And the phase response is $\angle H(\omega) = atan(-0.25sin\omega, 1 - 0.25cos\omega)$.

c) Sketch the magnitude and phase responses of the system. Please make sure to clearly label your axes.

Question 3 (filters)
This question studies a filter given by the following transfer function,

$$H(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}.$$

a) In this part, you may assume that the parameter $a$ is 0.5, i.e., $a = 0.5$.

- Determine where the filter has poles and/or zeros.
- Sketch a corresponding pole-zero diagram. As before, please make sure to clearly label your axes.

**Solution:** Substituting $a = 0.5$, the filter can be rewritten as follows,

$$H(z) = \frac{1-0.5}{2}\frac{1+z^{-1}}{1-0.5z^{-1}} = 0.25\frac{z+1}{z-0.5},$$

and it can be seen that there is a zero at $z_1 = -1$ and pole at $p_1 = +0.5$.

b) What type of filter is this (high pass, low pass, band stop, etc.)? Make sure to justify your answer.

**Solution:** The zero at $z_1 = -1$ corresponds to frequency $\omega = \pi$ on the unit circle, and it attenuates larger frequencies. On the other hand, the pole at $p_1 = +0.5$ corresponds to frequency $\omega = 0$, and it magnifies lower frequencies. Therefore, our filter is a low pass.

c) In this part, you need to compute the value of $a$ to ensure that $|H(\frac{\pi}{2})|^2 = \frac{1}{2}$.

**Solution:** Note that $\omega = \pi/2$ corresponds to $z = e^{j\omega} = e^{j\pi/2} = +j$. Therefore,

$$H(z = +j) = \frac{1-a}{2}\frac{1+j^{-1}}{1-aj^{-1}} = \frac{1-a}{2}\frac{1-j}{1+aj},$$

and the squared magnitude is

$$|H(z = +j)|^2 = \left(\frac{1-a}{2}\right)^2 \frac{1^2+1^2}{1^2+a^2} = \frac{2(1-a)^2}{4(1+a^2)},$$

because a complex number $a + bj$ has squared magnitude $a^2 + b^2$. We are given that $|H(z = +j)|^2 = \frac{1}{2}$, implying that $(1-a)^2 = 1 - 2a + a^2 = 1 + a^2$, which shows that $a = 0$.

d) Derive a difference equation corresponding to the filter $H(z)$. (If you are unsure about the value you calculated for the parameter $a$ in Part c, you may assume that $a = 0.5$ as indicated in Part a.)
**Solution:** Using $a = 0$, we have $H(z) = Y(z)/X(z) = \frac{z+1}{4(z-0.5)}$, implying that $4Y(z)(z-0.5) = X(z)(z+1)$. Multiplying both sides of the equation by $z^{-1}$, $4Y(z)(1-0.5z^{-1}) = X(z)(1+z^{-1})$. Taking the inverse $z$-transform, $4y(n) - 2y(n-1) = x(n) + x(n-1)$.

Question 4 (Gibbs effect)
In this question, we will investigate the Fourier response of a finite duration high pass filter. We define the frequency response of an ideal high pass filter as follows,

$$H_{HP}(\omega) = \begin{cases} 1 & \omega_c < |\omega| \leq \pi \\ 0 & \text{else} \end{cases},$$

where $\omega_c$ is a cut off frequency.

a) Compute the infinite duration impulse response, $h_{HP}(n)$, that corresponds to the ideal high pass. One way to do so involves expressing our filter as $H_{HP}(\omega) = 1 - H_{LP}(\omega)$, where $H_{LP}(\cdot)$ is an ideal low pass. Consequently, $h_{HP}(n) = 1 - h_{LP}(n)$, and we have computed the impulses responses of ideal low pass filters before. (You are welcome to compute $h_{HP}(n)$ using some different technique, of course.)

**Solution:** There were several styles of solutions. A simple one notes that $h_{LP}(n)$ is a sinc function, and then adjusts to provide the high pass response.

b) To make $h_{HP}(n)$ implementable, we truncate it to indices between $-N$ and $+N$. The impulse response of this truncated filter is denoted by

$$h_{HP,N}(n) = \begin{cases} h_{HP}(n) & |n| \leq N \\ 0 & else \end{cases}.$$

Sketch the frequency response, $H_{HP,N}(\omega)$, of this truncated filter. As before, please make sure to clearly label your axes. (Hint: recall the Gibbs effect. You can assume that $N$ is large yet finite.)

**Solution:** We have a high pass filter that allows frequencies in the range $\omega_c < |\omega| \leq \pi$ pass through with magnitude response near 1, while frequencies below the cut off frequency, i.e., $|\omega| < \omega_c$, are blocked with a response near zero. That said, the Gibbs effect yields some ringing artifacts and especially overshoots near the transition between the pass band and stop band. Another interesting point is that the frequencies $\omega \in \{-\pi, +\pi\}$ are symmetric, meaning that $H(\pi) = H(-\pi)$. Therefore, there is no Gibbs style effect at that frequency.

c) Will $H_{HP,N}(\omega)$ converge to the ideal desired frequency response, $H_{HP}(\omega)$ as we increase $N$? Please make sure to justify your answer.

**Solution:** Owing to the Gibbs effect, the discontinuity in the frequency domain causes the response of the truncated filter to *not* have pointwise convergence to the ideal high pass filter. Instead, there will be overshoots and undershoots whose amplitude is roughly 0.09, which is 9% of the gap between the filter response in the stop and pass bands. That said, this region with overshoots will become narrower as $N$ is increased, meaning that we will have convergence in the mean square sense.