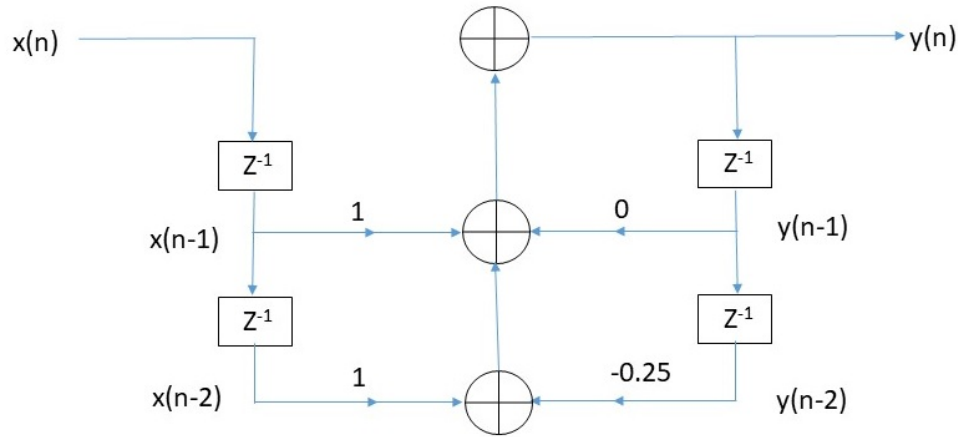


Question 1 (system response)

Consider the following block diagram.



(a) Provide a difference equation that relates the input,  $x(n)$ , and output,  $y(n)$ .

**Solution:** It can be seen from the figure that

$$y(n) = x(n - 1) + x(n - 2) - 0.25y(n - 2).$$

(b) Compute the transfer function,  $H(z) = Y(z)/X(z)$ , in the  $z$ -transform domain.

**Solution:** Taking the  $z$ -transform of the difference equation,

$$Y(z) = X(z)z^{-1} + X(z)z^{-2} - 0.25Y(z)z^{-2}.$$

Rearranging terms,

$$Y(z)(1 + 0.25z^{-2}) = X(z)(z^{-1} + z^{-2}).$$

The transfer function is

$$H(z) = Y(z)/X(z) = \frac{z^{-1} + z^{-2}}{1 + 0.25z^{-2}} = \frac{z + 1}{z^2 + 0.25}.$$

(c) Based on part (b) above, sketch the pole zero plot for  $H$ . Where are the poles and zeros located? (If you are unsure about your answer in part (b), you may use the transfer function  $\tilde{H}(z) = \frac{z^{-1}}{(z-0.5j)(z+0.5j)}$ .)

**Solution:** The poles are located at  $p_1 = -0.5j$  and  $p_2 = +0.5j$ . The single zero is located at  $z_1 = -1$ . A plot would show the real axis, imaginary axis, likely the unit circle, and the zeros (depicted with a circle) and poles (x marks).

Using the transfer function provided, the zero is at  $z_1 = +1$ .

(d) Based on part (c) above, what sort of filter is  $H$ ? Is it a low pass, band stop, high pass, band pass, other? Make sure to justify your answer.

**Solution:** With a zero at -1, the higher frequencies are blocked. Poles at  $\pm 0.5j$  accentuate frequencies near  $\omega = \pm 0.5\pi$ . At lower frequencies, the response is likely somewhat lower. This could be a low pass filter (LPF) or band pass filter (BPF).

Using the transfer function provided in part (c), the lower frequencies are blocked, and at higher frequencies the response is somewhat lower. This could be a high pass filter (HPF) or BPF.

(e) Based on the transfer function of part (b), compute and sketch the magnitude of the frequency response,  $|H(\omega)|$ , as a function of  $\omega \in [-\pi, +\pi]$ . In your sketch, make sure to label the axes properly and highlight the effects of zeros and poles on  $|H(\omega)|$ . (If you are unsure about part (b), you may use  $\tilde{H}(z)$  from (c).)

**Solution:** The transfer function is

$$H(\omega) = H(z = e^{j\omega}) = \frac{e^{j\omega} + 1}{e^{2j\omega} + 0.25}.$$

As described previously,  $|H(\omega)|$  peaks at  $\omega = \pm 0.5\pi$ , is zero at  $\omega = \pi$ , and somewhat low at lower frequencies near  $\omega = 0$ .

Using the transfer function provided in part (c), the transfer function is

$$H(\omega) = H(z = e^{j\omega}) = \frac{e^{j\omega} - 1}{e^{2j\omega} + 0.25}.$$

As described previously,  $|H(\omega)|$  peaks at  $\omega = \pm 0.5\pi$ , is zero at  $\omega = 0$ , and somewhat low at higher frequencies near  $\omega = \pi$ .

Question 2 (continuous time Fourier transform)

The Fourier transform of a continuous time aperiodic signal,  $x_a(t)$ , is  $X(F) = e^{-|F|}$ , where  $|F|$  denotes the absolute value of  $F$ . You will compute  $x_a(t)$  in several steps.

(a) The Fourier transform  $X(F)$  can be expressed as a sum,

$$X(F) = X_1(F) + X_2(F), \quad (1)$$

where

$$X_1(F) = e^{-F}u(F), \quad X_2(F) = e^F u(-F),$$

and  $X_1$  and  $X_2$  are Fourier responses for positive and negative frequencies, respectively. Compute  $x_1(t)$ , the continuous time aperiodic signal that corresponds to  $X_1(F)$ , using the inverse transform,  $x_1(t) = \int_{F=-\infty}^{+\infty} X_1(F)e^{+j2\pi Ft}dF$ .

**Solution:**

$$\begin{aligned} x_1(t) &= \int_{F=-\infty}^{+\infty} X_1(F)e^{+j2\pi Ft}dF \\ &= \int_{F=-\infty}^{+\infty} e^{-F}u(F)e^{+j2\pi Ft}dF \\ &= \int_{F=0}^{+\infty} e^{-F}e^{+j2\pi Ft}dF \\ &= \int_{F=0}^{+\infty} e^{F(-1+j2\pi t)}dF \\ &= \left. \frac{e^{F(-1+j2\pi t)}}{-1+j2\pi t} \right|_0^{+\infty} \\ &= \frac{0-1}{-1+j2\pi t} \\ &= \frac{1}{1-j2\pi t}. \end{aligned}$$

(b) Compute  $x_2(t)$ . (Hint:  $X_2(F)$  is a reversed version of  $X_1(F)$ . Although we didn't discuss properties of continuous time transforms in class, you should be able to guess the relation between  $x_1(t)$  and  $x_2(t)$ , allowing you to "double check" your calculation.)

**Solution:** A similar computation results in

$$x_2(t) = \frac{1}{1+j2\pi t}.$$

This computation makes sense, because we would anticipate that  $x_2(t)$  is the complex conjugate of  $x_1(t)$ .

(c) Based on linearity of the Fourier transform and equation (1) on page 3, compute  $x_a(t)$  by combining your results for  $x_1(t)$  and  $x_2(t)$ .

**Solution:** We sum  $X_1(t)$  and  $x_2(t)$ ,

$$\begin{aligned}x(t) = x_1(t) + x_2(t) &= \frac{1}{1 - j2\pi t} + \frac{1}{1 + j2\pi t} \\ &= \frac{[1 + j2\pi t] + [1 - j2\pi t]}{[1 + j2\pi t][1 - j2\pi t]} \\ &= \frac{2}{1^2 - (j2\pi t)^2} \\ &= \frac{2}{1 + 4\pi^2 t^2}.\end{aligned}$$

Question 3 (filter design considerations)

Consider a 3-tap filter of the form  $h = [h(0), h(1), h(2)]$ , where the underline corresponds to time index  $n = 0$ . This filter  $h$  satisfies:

- For a step function input,  $x_1(n) = u(n)$ , the output  $y_1 = h * x_1$  at time zero satisfies  $y_1(0) = 2$ . (Hint: this provides information on  $h(0)$ .)
- The DC component of the input is magnified at the output by a factor of 3. (Hint: this provides information on  $H(\omega = 0)$ .)
- An input,  $x_2(n) = (-1)^n$ , results in an output,  $y_2(n) = 5x_2(n)$ . (Hint: this provides information on  $H(\omega = \pi)$ .)

(a) What is the impulse response,  $h$ ?

**Solution:** The first condition,  $y_1(0) = 2$ , informs us that  $y_1(0) = x_1(0)h(0) = 1 \cdot h(0)$ . Therefore,  $h(0) = 2$ . The second condition,  $H(0) = 3$ , informs us that  $h(0) + h(1) + h(2) = 3$ . The third condition,  $H(\pi) = 5$ , informs us that  $h(0) - h(1) + h(2) = 5$ . Combining  $h(0) + h(1) + h(2) = 3$  and  $h(0) - h(1) + h(2) = 5$ , we see that  $2h(1) = 3 - 5 = -2$ , and so  $h(1) = -1$ . Next,  $h(0) + h(1) + h(2) = 2 + (-1) + h(2) = 3$ , and so  $h(2) = 2$ . We conclude that  $h = [\underline{2}, -1, 2]$ .

(b) Compute the Fourier response,  $H(\omega)$ , and its magnitude,  $|H(\omega)|$ . (If you are unsure about your answer in part (a), you may assume  $h = [\underline{3}, -2, 3]$ .)

**Solution:** Using the correct solution,  $H(\omega) = 2 - 1e^{-j\omega} + 2e^{-2j\omega}$ . We can express  $H(\omega)$  as

$$H(\omega) = e^{-j\omega}[2e^{+j\omega} - 1 + 2e^{+j\omega}] = e^{-j\omega}[4 \cos(\omega) - 1],$$

and the magnitude is  $|H(\omega)| = |4 \cos(\omega) - 1|$ .

Using the impulse response  $h$  provided,  $H(\omega) = 3 - 2e^{-j\omega} + 3e^{-2j\omega}$ . We can express  $H(\omega)$  as

$$H(\omega) = e^{-j\omega}[3e^{+j\omega} - 2 + 3e^{+j\omega}] = e^{-j\omega}[6 \cos(\omega) - 2],$$

and the magnitude is  $|H(\omega)| = |6 \cos(\omega) - 2|$ .

Question 4 (MATLAB)

The following MATLAB code computes the Fourier transform of a signal and plots its magnitude and phase. There are missing parts on Lines 3, 5, and 6 below.

```
1. t=0.01*(1:256); % time domain
2. m=2*cos(20*p*t); % signal
3. M=_____; % compute Fourier response
4. f=1:256; % frequencies
5. subplot(2,1,1), plot(f,_____); % plot magnitude of Fourier
6. subplot(2,1,2), plot(f,_____); % -- phase --
```

(a) What is the missing code on Line 3?

**Solution:** `fft(m)`

(b) What is the missing code on Line 5?

**Solution:** `abs(M)`

(c) What is the missing code on Line 6?

**Solution:** `angle(M)`

Question 5 (Gibbs effect)

We learned that a discrete time aperiodic signal,  $x(n)$ , may have a Fourier transform  $X(\omega)$  with discontinuities. In this case,  $X_N(\omega)$ , the transform based on truncating the signal to the range  $n \in \{-N, \dots, +N\}$  may converge to  $X(\omega)$  in a mean square sense but not a pointwise sense. The lack of pointwise convergence is due to the signal  $x(n)$  not being absolutely summable. Consider instead a continuous time signal,  $x(t)$ , that has period  $T_p = 1$ , and within the time interval  $t \in (-0.5, +0.5)$  satisfies

$$x(t) = \begin{cases} 1 & |t| < 0.1 \\ 0 & \text{else} \end{cases}.$$

The Fourier series of  $x(t)$  is  $x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+j2\pi kt/T}$ , and based on our derivation in class (page 7 of slides for Chapter 4) we saw that

$$C_k = 0.2 \frac{\sin(0.2\pi k)}{0.2\pi k} = 0.2 \text{sinc}(0.2\pi k).$$

Define a signal derived from a truncated Fourier series,

$$x_N(t) = \sum_{k=-N}^{+N} C_k e^{+j2\pi kt/T}.$$

(a) Explain why  $x_N(t)$  does not converge pointwise to  $x(t)$  by evaluating whether the Fourier coefficients,  $(C_k)_{k=-\infty}^{+\infty}$ , are absolutely summable.

**Solution:** The  $C_k$  coefficients decay as  $1/k$ , and an infinite summation,  $\sum_k 1/k$  is well known to diverge. Therefore, the sequence is not absolutely summable, and so  $x_N(t)$  may not converge pointwise to  $x(t)$ .

(b) Nonetheless,  $x_N(t)$  converges in the mean square sense to  $x(t)$ . Show this by using the Parseval relation to shift the discussion to mean square convergence of the Fourier coefficients,  $(C_k)_{k=-\infty}^{+\infty}$ , and evaluate their energy.

**Solution:** The Parseval relation tells us that the total energy of all  $C_k$  coefficients equals the total energy of  $x(t)$ , which is clearly finite. Therefore, the energy removed when truncating some measurements is finite, and when  $N$  is increased the energy in the error between  $x_N(t)$  and  $x(t)$  vanishes.