ECE 421 Introduction to Signal Processing

Midterm 2 – Spring 2021

March 15, 2021

Please remember to justify your answers carefully.

Last name: ____________________ First name: ____________________

Please recall the course academic integrity policy for tests:
No cooperation or "collaboration" between students is allowed. Especially during an online course experience, it could be tempting to text or email a friend. This is not allowed. You will be allowed to use your notes, books, a browser, and software such as Matlab.[1] However, while working on the test you should not text, email, or communicate with other people (certainly not other students) in any way, unless you are consulting with the course staff. **By submitting the test, you will be acknowledging that you completed the work on your own without the help of others in any capacity.** Any such aid would be unauthorized and a violation of the academic integrity policy.

[1] You can use the browser to access Moodle, the course webpage, and look up technical topics. Similar to a normal test, you must not communicate with other people.

Question 1 (continuous time Fourier transform)
Consider a continuous time aperiodic signal,

$$x_a(t) = e^{-3t}u(t+2) + e^{-4t}u(t-2).$$

You will compute the Fourier transform of $x_a(t)$ in several steps.

(a) The continuous time aperiodic signal $x_a(t)$ can be expressed as a sum,

$$x_a(t) = x_{a1}(t) + x_{a2}(t), \tag{1}$$

where $x_{a1}(t) = e^{-3t}u(t+2)$ and $x_{a2}(t) = e^{-4t}u(t-2)$. Compute $X_1(F)$, the Fourier transform that corresponds to $x_{a1}(t)$, using the transform, $X_1(F) = \int_{t=-\infty}^{\infty} x_{a1}(t)e^{-j2\pi Ft}dt$.

(b) Compute $X_2(F)$, the Fourier transform that corresponds to $x_{a2}(t)$.

(c) Based on linearity of the Fourier transform and equation (1) on Page 2, compute $X(F)$ by combining your results for $X_1(F)$ and $X_2(F)$

Question 2 (properties of Fourier transforms)
Consider a discrete time aperiodic signal, $x(n) = \{2, 1, 0, \underline{1}, -2, 3, 2\}$, where the underline corresponds to time index $n = 0$. Compute the following quantities without explicitly computing $X(\omega)$, the Fourier transform of $x(n)$.

(a) $X(0)$.

(b) $X(\pi)$.

(c) $\int_{-\pi}^{\pi} X(\omega)e^{-jw}d\omega$.

(d) $\int_{-\pi}^{\pi} \mid X(\omega) \mid^2 d\omega$.

Question 3 (discrete time Fourier transform)
In this question, you will compute the Fourier transform of the following discrete time signal,

$$x(n) = \begin{cases} a^{|n|}\cos(\theta n) & |n| \le N \\ 0 & \text{else} \end{cases},$$

where $0 < a < 1$. The computation can be simplified: (*i*) instead of the two-sided $|n| \le N$ in $x(n)$, we will consider a one-sided $0 \le n \le N$, i.e., $|n| = n$; (*ii*) instead of $\cos(\theta n)$, we can use its components, $\exp(j\theta n)$ and $\exp(-j\theta n)$; and (*iii*) we can initially remove these complex exponents.

(a) Consider a simplified version of $x(n)$,

$$x_1(n) = \begin{cases} a^n & 0 \le n \le N \\ 0 & \text{else} \end{cases}.$$

Compute $X_1(\omega)$, the Fourier transform of $x_1(n)$.

(b) We modulate $x_1(n)$ by a complex exponential,

$$x_2(n) = \begin{cases} a^n \exp(j\theta n) & 0 \leq n \leq N \\ 0 & \text{else} \end{cases}.$$

Compute $X_2(\omega)$, the Fourier transform of $x_2(n)$.

(c) Instead of the complex exponent $\exp(j\theta n)$ of $x_2(n)$, we use a cosine,

$$x_3(n) = \begin{cases} a^n \cos(\theta n) & 0 \leq n \leq N \\ 0 & \text{else} \end{cases}.$$

Compute $X_3(\omega)$, the Fourier transform of $x_3(n)$. To do so, recall that

$$\cos(\theta n) = \frac{1}{2}\exp(j\theta n) + \frac{1}{2}\exp(-j\theta n),$$

and utilize linearity of the transform. (If you are unsure about your answer in part (b), you may assume that $X_2(\omega) = \frac{a}{a\theta+\omega}$.)

(d) So far we have considered a one-sided signal, meaning that $0 \leq n \leq N$. However, our original signal $x(n)$ from Page 6 used a two-sided $|n| \leq N$. You will now compute the transform over $-N \leq n \leq 0$,

$$x_4(n) = \begin{cases} a^{-n}\cos(\theta n) & 0 \leq n \leq N \\ 0 & \text{else} \end{cases},$$

where we note that $a^{|n|} = a^{-n}$ for this range of n. Compute $X_4(\omega)$, the Fourier transform of $x_4(n)$. (If you are unsure about your answer in part (c), you may assume that $X_3(\omega) = \frac{a+\cos(\theta)}{a^2+\omega}$.)

Note: after you submit the test, you can complete the derivation of $X(\omega)$. For now, submitting $X_1(\omega)$, $X_2(\omega)$, $X_3(\omega)$, and $X_4(\omega)$ is fine.