ECE 421 Introduction to Signal Processing

Test 5 – Spring 2021

May 3, 2021

Question 1 (Linear convolution via DFT; 6 points.)
Let $x(n) = \{\underline{6}, 2, 0, 3\}$ be the input sequence to a finite impulse response (FIR) filter $h(n) = \{\underline{1}, 2\}$ (the underline corresponds to time zero). In this question, we will look at linear convolution using zero padding.

(a) Compute the linear convolution, $y(n)$, using the time domain formula, i.e., $y(n) = \sum_m x(m)h(n-m)$.

Solution:

$$\begin{aligned} y(n) &= 1 \times \{\underline{6}, 2, 0, 3\} + 2 \times \{\underline{0}, 6, 2, 0, 3\} \\ &= \{\underline{6+0}, 2+12, 0+4, 3+0, 0+6\} \\ &= \{\underline{6}, 14, 4, 3, 6\}. \end{aligned}$$

(b) How many zeros need to be padded to $x(n)$ and $h(n)$, respectively, in order to avoid aliasing in the output, $y = x * h$, if we want to use the discrete Fourier transform (DFT) and inverse DFT (IDFT) to calculate $y(n)$? (Make sure to provide two numbers, one for x the other for h.)

Solution: The length of the convolution is $len(x)+len(h)-1 = 4+2-1 = 5$. Therefore, x and h must be zero padded using $5-4=1$ and $5-2=3$ zeros, respectively.

(c) What are $\tilde{h}$ and $\tilde{x}$ after zero padding h and x?

Solution: In light of part (b), the zero padded versions are $\{\underline{6}, 2, 0, 3, 0\}$ and $\{\underline{1}, 2, 0, 0, 0\}$, respectively.

(d) Suppose that x and h have been defined in Matlab as row vectors. The following Matlab code is supposed to calculate y(n). There are missing parts below. Please write down the missing parts in Lines 3, 5 and 8.

```
1.Lx=length(x);
2.Lh=length(h);
3.Ly= __;
4.xpad=[x zeros(1,Ly-Lx)];
5.hpad=__;
6.xf=fft(xpad);
7.hf=fft(hpad);
8.yf=__;
9.y=ifft(xf);
```

Solution: Line 3 is $Lx + Lh - 1$, Line 5 is $[h\ zeros(1, Ly - Lh)]$, and Line 8 is $xf. * hf$.

Question 2 (DFT properties; 8 points.)
Consider the system below:
A real-valued discrete time signal, $x(n)$, is of length $N = 6$. We are given several values of its discrete Fourier transform (DFT):

$$X(0) = 5, \quad X(1) = 2 + j, \quad X(2) = 1 + 2j, \quad X(3) = 4.$$

(a) Determine the missing values of $X(k)$ for $k \in \{4, 5\}$.

Solution: Because x is real valued, $X(4) = X(2)^* = 1 - 2j$ and $X(5) = X(1)^* = 2 - j$.

(b) We are given another length-6 discrete time signal, $y(n)$, whose DFT, $Y(k)$, is related to $X(k)$ as follows,

$$Y(k) = X(k)[0.5 + 0.5e^{j\pi k}].$$

Compute $Y(k)$ for $k \in \{0, 1, 2, 3, 4, 5\}$. (If you are not sure about your answers in part (a), you may assume that $X(4) = 35j$ and $X(5) = 5 + 2j$.)

Solution: The calculations are simplified by realizing that $[0.5 + 0.5e^{j\pi k}]$ is 1 for even k, else 0. We have $Y(0) = 5$, $Y(1) = 0$, $Y(2) = 1 + 2j$, $Y(3) = 0$, $Y(4) = 1 - 2j$, and $Y(5) = 0$.

(c) Consider the DFT time shift property, $x(n - m)_N \leftrightarrow X(k)e^{-j2\pi km/N}$, where $x(n - m)_N$ denotes time indices that are cyclical with respect to modulo N. Use the time shift property to express $y(n)$ in terms of $x(n)$. (We expect a response such as $y(n) = x(n) - 10$ or $y(n) = x(n+1)_N - x(n-2)_N$.)

Solution: Because $N = 6$ and $e^{j\pi k} = e^{j2\pi\frac{3k}{6}}$, the shift is $m = -3$ (identical to $m = +3$), $y(n) = 0.5x(n) + 0.5x(n - 3)_6$.

(d) Find $\sum_{n=0}^{5} x(n)x(n)^*$.

Solution: Using Parseval,

$$\begin{aligned}
\sum_n x(n)x(n)^* &= \frac{1}{N}\sum_k X(k)X(k)^* \\
&= \frac{1}{6}[5^2 + 2|2 + j|^2 + 2|1 + 2j|^2 + 4^2] \\
&= \frac{1}{6}[25 + 2(4 + 1) + 2(1 + 4) + 16] \\
&= \frac{1}{6}[25 + 10 + 10 + 16] \\
&= \frac{61}{6}.
\end{aligned}$$

Question 3 (IIR filter design; 6 points.)
Consider an analog filter with system function

$$H_a(s) = \frac{s+2}{s^2+4s+13}.$$

(a) Convert the analog filter into a digital infinite impulse response (IIR) filter $H(z)$ using a bilinear transformation, where the digital filter is supposed to have a resonant frequency of $\omega_r = \frac{\pi}{3}$.

Solution: We can see that the denominator has the form $s^2 + 4s + 13 = (s^2 + 4s + 4) + 9$, which is the square of the numerator plus 9. Therefore, $\Omega_r = \sqrt{9} = 3$. Because $\omega = 2\tan^{-1}(\Omega T/2)$, we have $\tan(\omega/2) = \Omega T/2$. Note that $\omega/2 = \frac{\pi}{6}$, and its tangent is $1/\sqrt{3}$. Therefore, $\Omega T/2 = 1/\sqrt{3}$, and $T = \frac{2}{\Omega\sqrt{3}} = \frac{2}{\sqrt{27}}$. Next, the relation between s and z is,

$$s = \frac{2(z-1)}{T(z+1)} = \sqrt{27}\frac{z-1}{z+1}.$$

Finally, we substitute s into $H(s)$,

$$H(z) = H\left(s = \sqrt{27}\frac{z-1}{z+1}\right) = \frac{\sqrt{27}\frac{z-1}{z+1} + 2}{\left(\sqrt{27}\frac{z-1}{z+1} + 2\right)^2 + 9}.$$

(b) In order to simplify the calculation, in the rest of question 3 please use a digital IIR filter with tranfer function

$$H(z) = \frac{(z+1)(z-1)}{z^2 - 1.5z - 1}$$

Where are the zero(s) and pole(s) of H(z)?

Solution: Zeros are roots of the numerator, $z_1 = -1$ and $z_2 = +1$. Poles are roots of the denominator, $z^2 - 1.5z - 1 = (z-2)(z+0.5)$, and so $p_1 = 2$ and $p_2 = -0.5$.

(c) Suppose that this system is stable. What is the region of convergence (ROC) of this filter? Is $H(z)$ a causal system?

Solution: BIBO stable means that the ROC includes the unit circle. Pole $p_1 = 2$ is outside the unit circle, and pole $p_2 = -0.5$ is inside the unit circle. Therefore, $ROC = \{0.5 < |z| < 2\}$. The first pole has an anti-causal response, and the second pole has a causal one. Therefore, no, $H(z)$ is not causal.