ECE 421
Introduction to Signal Processing

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Arithmetic and Geometric Series

Arithmetic series

- Length-N seqence of form Δ, 2Δ ,…, NΔ
- Example: 2, 4, 6, 8 (Δ=2, N=4)
- Claim: $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

Proof of arithmetic series sum

- How do we prove $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$?

- Proof1: By induction
  - Basis case: N=2, $\sum_{n=1}^{N} n = 1 + 2 = 3$, $\frac{N(N+1)}{2} = \frac{2(2+1)}{2} = 3$
  - Inductive step: assume for N, prove for N+1
    - $\sum_{n=1}^{N+1} n = [\sum_{n=1}^{N} n] + (N + 1) = \frac{N(N+1)}{2} + (N + 1) = \frac{(N+2)(N+1)}{2}$

- Proof2: Well-known mathematician Gauss used trick

Geometric series

- Length-N sequence of form A^k, A^{k+1}, …, A^{k+N-1}
- What's the sum?

- Define two *infinite sums*: $\Psi_1=1+A+A^2+\ldots$, $\Psi_2=A+A^2+A^3+\ldots$
  - $\Psi_2= A\Psi_1$ (each element in sequence multiplied by A)
  - $\Psi_2+1=\Psi_1$
- Conclusion: $A\Psi_1+1=\Psi_1$ → $1=\Psi_1(1-A)$ → $\Psi_1=\frac{1}{1-A}$
  - Caveat: $|A|<1$, else the summation diverges

- Example: $1+\frac{1}{2}+\frac{1}{4}+\cdots$
- $A=\frac{1}{2}$ → $1+\frac{1}{2}+\frac{1}{4}+\cdots = \frac{1}{1-A}=\frac{1}{1-\frac{1}{2}}=2$

Finite sums

- Recall $A^k, A^{k+1}, \ldots, A^{k+N-1}$
- This sequence is finite
- $\sum_{n=0}^{N-1} A^{k+n} = \sum_{n=0}^{N-1} A^k A^n = A^k \sum_{n=0}^{N-1} A^n$
- $\sum_{n=0}^{N-1} A^n = [\sum_{n=0}^{\infty} A^n] - [\sum_{n=N}^{\infty} A^n] = [\sum_{n=0}^{\infty} A^n](1-A^N)$
- In summary, $\sum_{n=0}^{N-1} A^{k+n} = Ak\,\frac{1-A^N}{1-A}$

Real-world example

- ECE421 student graduates & gets nice job
- Every year:
  - They save & invest $10,000
  - Previous investments grow by factor 1+p (p=7% growth is plausible)

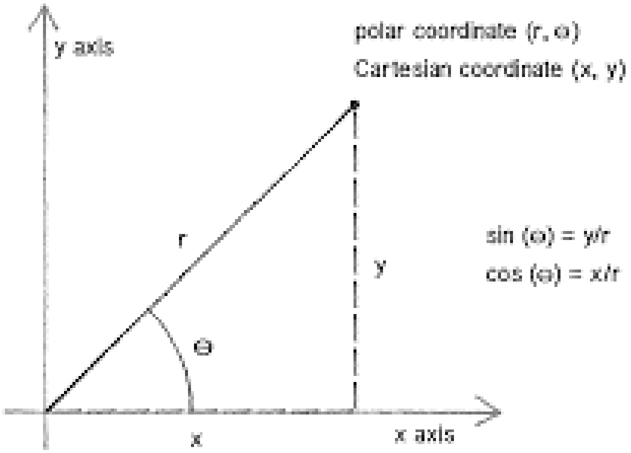
- After N years:
  - $10,000 from year 1 multiplied by $(1+p)^{N-1}$
  - From year 2, multiplied by $(1+p)^{N-2}$
  - Finally, from year N it's multiplied by $(1+p)^{N-N}=1$
- $\sum_{n=1}^{N}(1+p)^{N-n} = \sum_{n`=0}^{N-1}(1+p)^{n`} = \frac{1-(1+p)^N}{1-(1+p)}$
- Take p=7% & N=20 years, $\frac{1-(1+p)^N}{1-(1+p)} = \frac{-2.87}{-0.07} = 41$→ $410,000

Complex Numbers

Basics

- Unit imaginary number i (or j) satisfies $j^2=i^2=-1$

- Can consider complex number c in
  - Cartesian coordinates: $c=a+j\cdot b$
  - Polar coordinates: $c=r\cdot\exp(j\theta)$

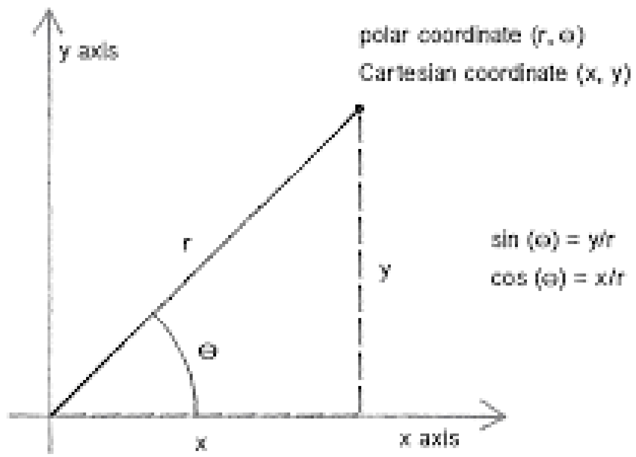


Details for polar coordinates

- Unit imaginary number i (or j) satisfies $j^2=i^2=-1$

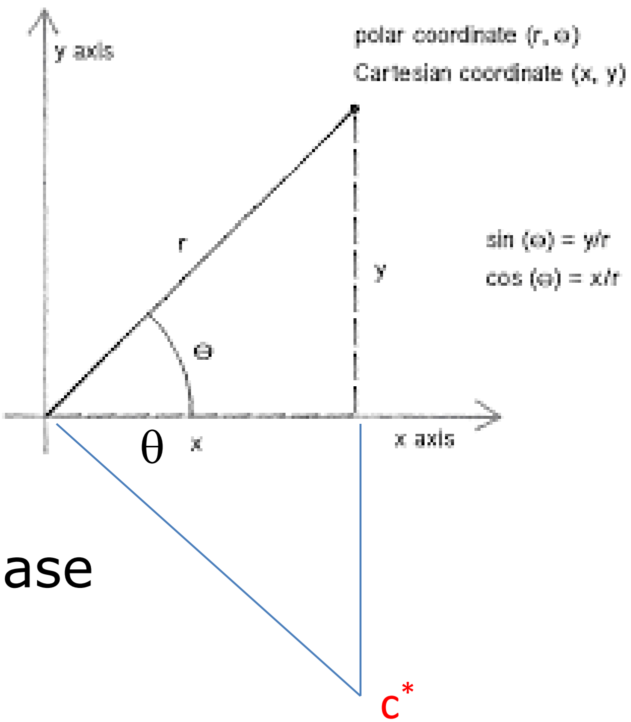
- Can consider complex number c in
  - Cartesian coordinates: $c=a+j\cdot b$
  - Polar coordinates: $c=r\cdot \exp(j\theta)$

- Details:
  - Radius r satisfies $r^2=a^2+b^2$
  - $a=r\cdot\cos(\theta)$
  - $b=r\cdot\sin(\theta)$



Complex conjugate

- Unit imaginary number i (or j) satisfies $j^2=i^2=-1$
- Can consider complex number c in
  - Cartesian coordinates: $c=a+j\cdot b$
  - Polar coordinates: $c=r\cdot\exp(j\theta)$
- Defined as $c^*=a-j\cdot b$
- Recall cosine/sine even/odd
  - $\cos(-\theta)=\cos(\theta)$, $\sin(-\theta)=-\sin(\theta)$
- Conclusion:
  - c^* has same radius r but flipped phase
  - $\angle\{c^*\}=-\angle\{c\}$



Euler’s formula

- Euler’s formula relates exponents and imaginary numbers
- $\exp(j\theta)=\cos(\theta)+j\cdot\sin(\theta)$

- Leonhard Euler 18th century mathematician
- Rationale: Taylor series
  - $\exp(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots$
  - $\exp(jx)=1+jx-\frac{1}{2!}x^2-\frac{j}{3!}x^3+\ldots$
  - $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$
  - $\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$

Powers of complex numbers

- Consider $c=r\cdot\exp(j\theta)$
- What is c^n?

- $c^n = (r\cdot\exp(j\theta))^n$

  $=r^n\cdot\exp(jn\theta)$ // $(\exp(x))^n=\exp(nx)$

  $=r^n\cdot[\cos(n\theta)+j\cdot\sin(n\theta)]$ // Euler's formula

Sines and cosines

- Consider $c=\exp(j\theta)=\cos(\theta)+j\cdot\sin(\theta)$

- And its conjugate $c^*=\exp(-j\theta)=\cos(\theta)-j\cdot\sin(\theta)$

- Sum / difference of c and c^*
  - $c+c^*=[\cos(\theta)+j\cdot\sin(\theta)]+[\cos(\theta)-j\cdot\sin(\theta)]=2\cos(\theta)$
  - $c-c^*=[\cos(\theta)+j\cdot\sin(\theta)]-[\cos(\theta)-j\cdot\sin(\theta)]=2j\cdot\sin(\theta)$

- Conclusion
  - $\cos(\theta)=\frac{1}{2}[\exp(j\theta)+\exp(-j\theta)]$
  - $\sin(\theta)=\frac{1}{2j}[\exp(j\theta)-\exp(-j\theta)]$

Products of cosines

- Various trigonometry identities can be derived:
  - $\cos(a)\cos(b)=0.5[\exp(ja)+\exp(-ja)]0.5[\exp(jb)+\exp(-jb)]$

    $=0.25[\exp(ja)\exp(jb)]+0.25[\exp(ja)\exp(-jb)]$
    $+0.25[\exp(-ja)\exp(jb)]+0.25[\exp(-ja)\exp(-jb)]$

    $=0.25\exp(j(a+b))+0.25\exp(j(a-b))$
    $+0.25\exp(j(-a+b))+0.25\exp(j(-a-b))$

    $=0.5[0.5\exp(j(a+b))+0.5\exp(-j(a+b))]$
    $+0.5[0.5\exp(j(a-b))+0.5\exp(-j(a-b))]$

    $=0.5\cos(a+b)+0.5\cos(a-b)$

Linear Algebra

Vectors and matrices

- Column vector $v_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$

- Row vector $v_2 = [b_1 \quad \ldots \quad b_N]$

- Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, more generally M rows, N columns
  - One row/column implies row/column vector

Inner products and matrix products

- Inner product between row and column vector:
  - $\begin{bmatrix} b_1 & \dots & b_N \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$=$a_1b_1$+…+$a_Nb_N$

- Matrix vector product AB: compute inner products between every row of A and every column of B

- Examples:
  - $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 1+2+3 & 1\cdot 4+2\cdot 5+3\cdot 6 \end{bmatrix} = \begin{bmatrix} 6 & 32 \end{bmatrix}$
  - $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 2 \\ 2\cdot 1+3\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Matrix inverse

- For matrix A such that y=Ax, inverse A^{-1} satisfies $A^{-1}y=x$
- Can show $AA^{-1}=A^{-1}A=I$
  - I identity matrix (ones on diagonal, else zero)
- Matrix (usually) invertible if it's square

- Example:
  - $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
  - $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$
  - $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 1/2 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 0 + 2 \cdot 1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{identity matrix}}$

Eigen-values and eigen-vectors

- For matrix A such that $Av=\lambda v$
  - λ called eigen-value
  - v eigen-vector

- There are algorithms for eigen-decomposition of a matrix

- Some properties
  - Trace of A (sum of elements on diagonal) equal to sum of e-values
  - Matrix A invertible if and only if all e-values nonzero