ECE 421
Introduction to Signal Processing

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Frequency-Domain Analysis of LTI Systems

Roadmap

- We have seen
  - Chapter 1 – from analog to digital and back
  - Chapter 2 – discrete time signals & systems; correlation
  - Chapter 3 – z-transforms; transfer functions; one-sided z
  - Chapter 4 – Fourier transforms
- *About to discuss Chapter 5*
  - LTI systems filter sinusoids; will compute transfer functions
  - Transient and steady state responses of LTI systems
  - Frequency response of rational systems
  - Filters including some simple examples

Sinusoids as Inputs of LTI Systems

[Reading material: Section 5.1]

Overview

- Have discussed that sinusoids (and complex exponentials) are eigen functions of LTI systems; will soon see:

$$x(n) = \cos(\omega n) \rightarrow \boxed{H} \rightarrow y(n) = |H(\omega)| \cos(\omega n + \Theta(\omega))$$

  - Magnification by $|H(\omega)|$
  - Phase shift $\Theta(\omega) = \angle\{H(\omega)\}$

$\angle\{H(\omega_k)\}$

- LTI systems are linear

$$x(n) = \Sigma_k c_k \cos(\omega_k n) \rightarrow \boxed{H} \rightarrow y(n) = \Sigma_k c_k |H(\omega_k)| \cos(\omega_k n + \Theta(\omega_k))$$

- Convenient to think about LTI systems in frequency domain; they are *filters*

Analysis for *one* complex exponential

- Recall that LTI system = convolution, $y(n)=\Sigma_k h(k)x(n-k)$
- Consider complex exponential input, $x(n)=e^{j\omega n}$

- What's the output?
- $y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$

$=\sum_{k=-\infty}^{+\infty} h(k)e^{j\omega(n-k)}$

$=\sum_{k=-\infty}^{+\infty} h(k)e^{j\omega n}e^{-j\omega k}$

$=e^{j\omega n}\left[\sum_{k=-\infty}^{+\infty} h(k)\, e^{-j\omega k}\right]$

- Red font part is Fourier transform, $H(\omega)=\sum_{k=-\infty}^{+\infty} h(k)\, e^{-j\omega k}$
  - $y(n)=e^{j\omega n}H(\omega)$
  - Input is eigen-vector; $H(\omega)$ eigen-value

Example 5.1

- Consider system H with impulse response $h(n)=(1/2)^n u(n)$
- Input $x(n)=e^{j\pi n/2}$
- Let's compute output y(n)

- Will show $H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

- Input corresponds to $\omega=\pi/2$
- Response is $H\left(\omega = \frac{\pi}{2}\right) = \frac{1}{1-\frac{1}{2}e^{-j\pi/2}} = \frac{1}{1-\frac{1}{2}(-j)} = \frac{1}{1+\frac{j}{2}}$
- Output $y(n)=e^{j\omega n}H(\omega) = e^{j\pi n/2}\frac{1}{1+\frac{j}{2}}$

Example 5.1 - continued

- System responds differently at other frequencies

- Consider DC (ω=0)
- $H(0) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}e^{-j0}} = \frac{1}{1-\frac{1}{2}} = 2$
- Constant input, x(n)=A, becomes y(n)=2A

Analysis of real-valued sinusoids

- In practice, x, H, y will often be real-valued
- Recall $|H(\omega)|=|H(-\omega)|$ and $<\{H(\omega)\}=\Theta(\omega)=-\Theta(-\omega)$

- Take $x_1(n)=e^{+j\omega n}$ and $x_2(n)=e^{-j\omega n}$
- Corresponding outputs:
  - $y_1(n)=H(\omega)e^{+j\omega n}=|H(\omega)|e^{+j\Theta(\omega)}e^{+j\omega n}$
  - $y_2(n)=H(-\omega)e^{-j\omega n}=|H(-\omega)|e^{+j\Theta(-\omega)}e^{-j\omega n}=|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$
  - Can see that $y_1(n)=(y_2(n))^*$

Summing two conjugate exponents

- Form sine/cosine inputs using sums of $x_1(n)$, $x_2(n)$:
  - $x_3(n)=0.5x_1(n)+0.5x_2(n)=0.5e^{+j\omega n}+0.5e^{-j\omega n}=\cos(\omega n)$
  - $x_4(n)=\frac{1}{2j}x_1(n)-\frac{1}{2j}x_2(n)=\frac{1}{2j}e^{+j\omega n}-\frac{1}{2j}e^{-j\omega n}=\sin(\omega n)$

- Outputs are also sums (superpositions):
  - $y_3(n)=0.5y_1(n)+0.5y_2(n)$

    $=0.5|H(\omega)|e^{+j\Theta(\omega)}e^{+j\omega n}+0.5|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$

    $=|H(\omega)|\{0.5e^{+j[\Theta(\omega)+\omega n]}+0.5e^{-j[\Theta(\omega)+\omega n]}\}$

    $=|H(\omega)|\cos(\omega n+\Theta(\omega))$

  - Can also show $y_4(n)=|H(\omega)|\sin(\omega n+\Theta(\omega))$
    - *Extra credit for people who develop this on message board*

Here's what we've shown…

- A bunch of signals are eigen functions of LTI systems:

- Exponents $x(n) = e^{j\omega n}$ → H → $y(n) = |H(\omega)| e^{j(\omega n + \Theta(\omega))}$

- Cosines $x(n) = \cos(\omega n)$ → H → $y(n) = |H(\omega)| \cos(\omega n + \Theta(\omega))$

- Sines $x(n) = \sin(\omega n)$ → H → $y(n) = |H(\omega)| \sin(\omega n + \Theta(\omega))$

- *Any sinusoidal signal* (arbitrary phase Φ, amplitude A)

$x(n) = A \cdot \sin(\omega n + \Phi)$ → H → $y(n) = |H(\omega)|\ A \cdot \sin(\omega n + \Phi + \Theta(\omega))$

Beyond Superposition of Sinusoids

Example: Moving Average Filters

Active learning (Example 5.1.2)

- Consider moving average filter

$$y(n) = \frac{1}{3}(x(n-1) + x(n) + x(n+1))$$

- Corresponds to convolution with h={1/3,<u>1/3</u>,1/3}
- Will compute frequency response

a) What is H(ω)?

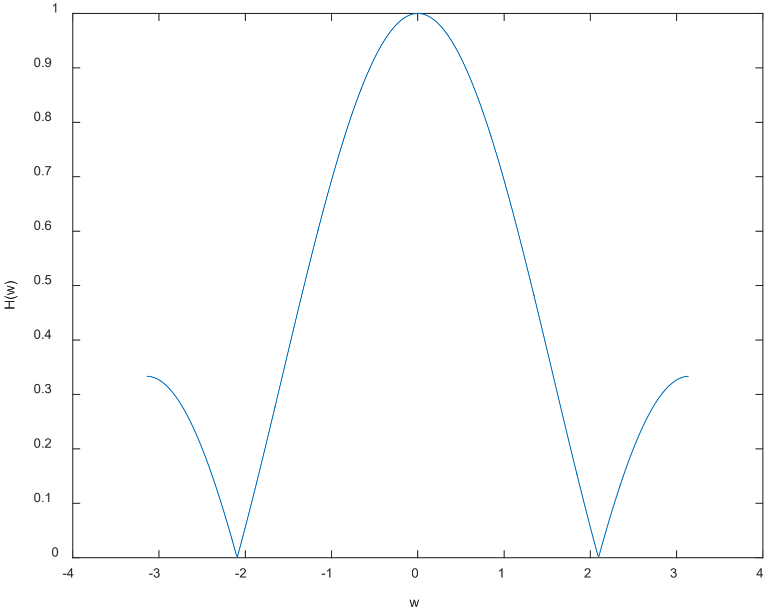
b) What is |H(ω)|? (Note real-valued H(ω).)

Example 5.1.2 continued

c) What is $\Theta(\omega)$?

d) Will plot in Matlab.

```
w=-pi:0.01:pi;
H=(exp(j*w)+1+exp(-j*w))/3;
plot(w,abs(H));
```



Are moving average filters useful?

- Moving average filters are lowpass filters
- Intuitively they smooth things by averaging adjacent samples

- We analyzed filter with 3 taps

- Based on previous class, $h(n) = \begin{cases} \frac{1}{2M+1}, & -M \leq n \leq +M \\ 0, & else \end{cases}$

- Has Fourier transform $H(\omega) = \frac{1}{2M+1} \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)}$
  - Also low pass
  - Peak at DC (ω=0)
  - M lobes
  - Width of main lobe proportional to 1/M → will block most noise

Example 5.1.4

- Consider system y(n)=ay(n-1)+bx(n), 0<a<1

a) Compute magnitude and phase of frequency response

  - In z domain: will show $H(z) = \frac{b}{1-az^{-1}}$
  - Convert to Fourier $H(\omega) = H\left(z = e^{j\omega}\right) = \frac{b}{1-ae^{-j\omega}} = \frac{b}{[1-\mathrm{a}*\cos(\omega)]+j[a*\sin(\omega)]}$
  - Magnitude: $|H(\omega)| = \frac{b}{\sqrt{[1-\mathrm{a}*\cos(\omega)]^2+[a*\sin(\omega)]^2}} = \frac{b}{\sqrt{1+a^2-2a*\cos(\omega)}}$
  - Phase can also be computed

b) Select constant b such that $\max_\omega |H(\omega)|=1$

  - Denominator minimal for $\omega=0$ → $H(0) = \frac{b}{\sqrt{1+a^2-2a}} = \frac{b}{1-a}$ → b=1-a

Are Sinusoids Useful?

Motivation

- We’ve seen how LTI systems respond to sinuosids

- Problem: Inputs of LTI systems are not sinusoidal in practice

- Is our situation hopeless?

From individual sinusoids to superpositions

- How about a *finite* superposition of sinusoids?
- Input: $x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \Phi_i)$
- Output: $y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| \cos(\omega_i n + \varphi_i + \Theta(\omega_i))$
- But finite superpositions of sinusoids have infinite duration
- Maybe still hopeless?

Infinite superpositions

- If Fourier exists (technical conditions?) can express input as *integration* over *infinitely* many sinusoids
- Input: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\omega)e^{+j\omega n}d\omega$
- Structure of output analogous to what we've seen

Transient and Steady State Response of LTI Systems

[Reading material: Section 5.1]

Motivation

- Suppose exponential signal begins at some point
  - Past – input was zero
  - Present – it starts
  - Future – exponent continues
- If system is BIBO, output will (eventually!) become exponential
  - Rationale: system LTI (linear) & BIBO → responses to transient stuff vanish (eventually!)
- Decay to zero of transient component is typically “fast enough”

Active learning (Example 5.1.5)

- LTI system has response $h(n)=(1/4)^n u(n)$
- Input $x(n)=(1/2)^n u(n)$
- Want to determine spectrum (Fourier transform) of output

a) What's the frequency response $H(\omega)$?

Example 5.1.5 continued

- $h(n)=(1/4)^n u(n)$, $x(n)=(1/2)^n u(n)$

b) Can also show $X(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

c) Spectrum of output $Y(\omega) = H(\omega)X(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \frac{1}{1-\frac{1}{4}e^{-j\omega}}$

– Can be simplified using partial fraction expansion, etc.

Frequency Response of Rational Systems

[Reading material: Section 5.2]

Why focus so much on rational?

- Difference equations have rational transfer functions
- Many practical systems use filters implemented with difference equations
- Useful to build intuition about rational transfer functions

Magnitude of rational transfer function

- Recall $H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k e^{-j\omega k}} = b_0 \frac{\prod_{k=1}^{M}(1-z_k e^{-j\omega})}{\prod_{k=1}^{N}(1-p_k e^{-j\omega})}$
  - Poles are p_k
  - Zeros are z_k

- Often interested in squared magnitude, $|H(\omega)|^2$
- $|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega)$

Example 5.2.1

- Difference equation, y(n)=-0.1y(n-1)+0.2y(n-2)+x(n)+x(n-1)
- Will compute $|H(\omega)|^2$

- Difference equation in z domain: $Y(z)[1+0.1z^{-1}-0.2z^{-2}]=X(z)[1+z^{-1}]$
- $H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}$
- $|H(\omega)|^2 = H(z)H(z^{-1}) = \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}} \frac{1+z}{1+0.1z-0.2z^2}$

$$= \frac{2+(z+z^{-1})}{1.05+0.08(z+z^{-1})-0.2(z^2+z^{-2})}$$

Example 5.2.1 continued

- Let's evaluate $|H(\omega)|^2 = \frac{2+(z+z^{-1})}{1.05+0.08(z+z^{-1})-0.2(z^2+z^{-2})}$
  - Note how $(z+z^{-1})$ and (z^2+z^{-2}) appear together
  - $z+z^{-1}=e^{j\omega}+e^{-j\omega}=2\cos(\omega)$ and $z^2+z^{-2}=2\cos(2\omega)$
  - Cosines are real valued → $|H(\omega)|^2$ real valued
- $|H(\omega)|^2 = \frac{2+(z+z^{-1})}{1.05+0.08(z+z^{-1})-0.2(z^2+z^{-2})} = \frac{2+2\cos(\omega)}{1.05+0.16\cos(\omega)-0.4\cos(2\omega)}$

- Magnitude squared $|H(\omega)|^2$ can be determined from H(z)
- Not vice versa

Another way to express rational

- Let's look at rational transfer functions another way
- $H(\omega) = b_0 \frac{\prod_{k=1}^{M}(1-z_k e^{-j\omega})}{\prod_{k=1}^{N}(1-p_k e^{-j\omega})} = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-z_k)}{\prod_{k=1}^{N}(e^{j\omega}-p_k)}$
- Rewrite terms corresponding to zeros/poles in *polar form* (magnitude & phase)
  - Useful because $e^{j\omega}$-*const* interpreted graphically (coming up)
  - Define $e^{j\omega}-z_k=V_k(\omega)e^{j\Theta_k(\omega)}$ and $e^{j\omega}-p_k=U_k(\omega)e^{j\Phi_k(\omega)}$
  - Phases of zeros/poles Θ_k / Φ_k
  - Magnitude of zeros/poles V_k / U_k

Phase and magnitude of transfer function

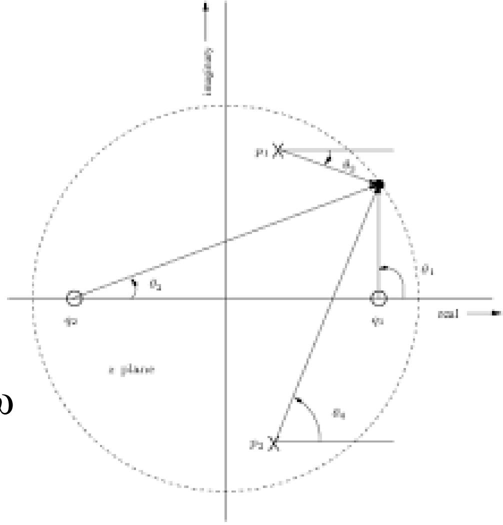
- Recall $H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-z_k)}{\prod_{k=1}^{N}(e^{j\omega}-p_k)}$

- Apply definitions for Θ_k, Φ_k, V_k, U_k

- Magnitude $|H(\omega)| = |b_0| \frac{V_1(\omega)V_2(\omega)\cdots V_M(\omega)}{U_1(\omega)U_2(\omega)\cdots U_N(\omega)}$

- Phase $< \{H(\omega)\} =< \{b_0\} + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega)$

Graphical interpretation



- Transfer function can be interpreted graphically
- Frequency ω corresponds to $z=e^{j\omega}$ on unit circle
- Consider vectors between zeros/poles and $z=e^{j\omega}$
- Increase ω → move counter-clockwise → Θ_k, Φ_k, V_k, U_k change
- Online video visualizes this nicely:
  - https://engineering.purdue.edu/VISE/ee438/demos/flash/pole_zero.html

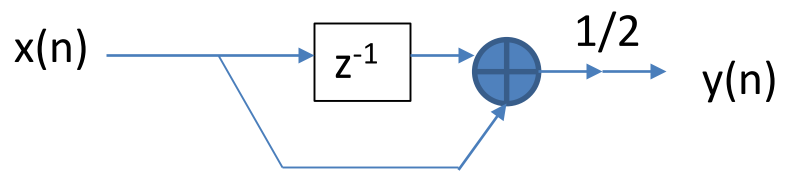
Example 5.2.2

- Consider LTI system, $H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$
- Will evaluate frequency response

- Zero at z=0, pole at p=0.8
- Frequency response, $H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.8}$
- Magnitude response
  - $|H(\omega)| = \frac{|e^{j\omega}|}{|e^{j\omega}-0.8|} = \frac{1}{|\cos(\omega)+jsin(\omega)-0.8|} = \frac{1}{\sqrt{(\cos(\omega)-0.8)^2+(\sin(\omega))^2}} = \frac{1}{\sqrt{cos^2(\omega)+sin^2(\omega)+0.64-1.6\cos(\omega)}} = \frac{1}{\sqrt{1.64-1.6\cos(\omega)}}$
- Phase $\Theta(\omega) =< \{e^{j\omega}\} -< \{e^{j\omega} - 0.8\} = \omega - tan^{-1}(\frac{\sin(\omega)}{\cos(\omega)-0.8})$

Another example (Question 5.10(a))

- Determine magnitude and phase response of following system



- $y(n)=0.5(x(n)+x(n-1))$
- $Y(z)=0.5X(z)(1+z^{-1})$
- $H(z) = \frac{Y(z)}{X(z)} = 0.5(1+z^{-1})$
- $H(\omega)=0.5+0.5e^{-j\omega}=0.5e^{-0.5j\omega}(e^{+0.5j\omega}+e^{-0.5j\omega})= e^{-0.5j\omega}\cos(0.5\omega)$
- $|H(\omega)|=|\cos(0.5\omega)|$
- $\Theta(\omega)=-0.5\omega+\angle\{\cos(0.5\omega)\}$
- Note that $\angle\{\cos(0.5\omega)\}$ is 0 (positive cosine) or π (negative)

Filters

[Reading material: Section 5.4]

What's a filter?

- Device that separates desirable / good from undesirable / bad

- Non-DSP examples:
  - Cars: air and oil filters let clean stuff pass through, block dirt
  - Photography: ultraviolet (UV) filter lets visible light pass though, blocks UV

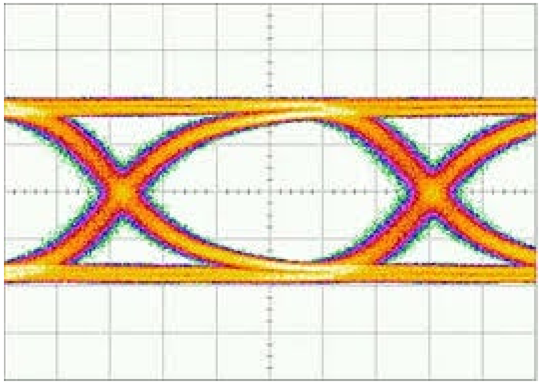
- Our focus is on discrete time LTI-system-based filters
  - Has frequency response $H(\omega)$
  - Filters signal such that $Y(\omega)=H(\omega)X(\omega)$
  - H *shapes* spectrum; passes lots of spectral content through some freqs while blocking others

Applications

- Filters appear in many applications
- Noise removal/denoising
  - y=x+z
  - Suppose that input x dominated by low freqs & noise z by high freqs
  - Natural choice of H blocks high freqs
- Will explore image denoising in Project 5
- Image corrupted by noise; improve w/ denoising algorithm

More applications

- Communications
  - Transmitted waveforms must adhere to strict spectral regulations (conserves electro magnetic bandwidth)
  - Spectral characteristics may make it difficult to separate input levels



- Matched filter improves separation between levels → simplifies decoding

And more

- Audio (Project 3)
  - Some spectral bands in audio (music) are more / less desirable to listen to
    - Different instruments could be in different bands
    - Noise is often broadband
  - Can improve user experience by enhancing/attenuating different freqs

- Correcting for imperfect analog hardware
  - Consider audio system
  - Analog loud speakers may distort music
  - Pre-processing the signal prior to the loud speaker (output stage) can improve overall listening experience
  - More flexible / cheaper / precise to do so digitally

What do we want filter to do?

- There are lots of types of LTI systems; want idealized properties
- Low complexity (simple, fast, low-memory, difference equation with few taps…)
- Low delay – 1 ms delay OK for phone conversation; 5 s not
- BIBO stable – unstable filter may cause (analog) hardware to mess up / saturate / even burn up (!)

Desirable frequency domain properties

- *Ideal* frequency response
  - Want to 100% pass through desirable frequencies
  - Want to 100% block undesirable

- Linear phase
  - Suppose signal bandlimited to $\omega_1<\omega<\omega_2$
  - Consider filter $H(\omega) = \begin{cases} ce^{-j\omega n_0}, \omega_1<\omega<\omega_2 \\ 0, \quad else \end{cases}$
  - Can show $y(n)=cx(n-n_0)$
    - Output delayed and amplified (not distorted)
  - Phase $\Theta(\omega)=-\omega n_0$ linear in ω
  - Phase that isn't linear distorts signal → prefer linear phase

Desirable frequency domain properties

- Ideal frequency response
  - Want to 100% pass through desirable frequencies
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- *Linear phase*
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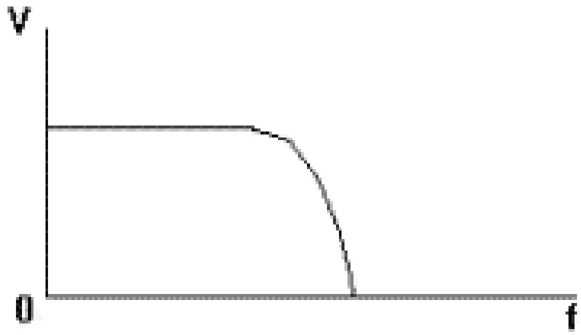
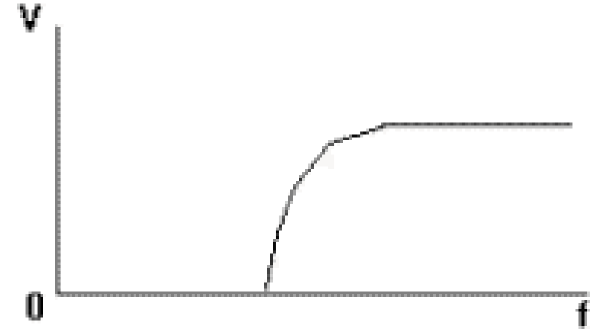
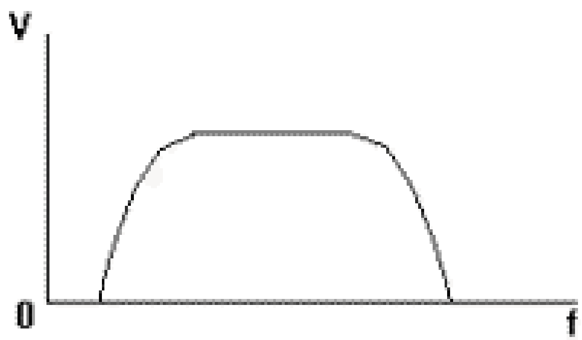
Typical desired frequency responses



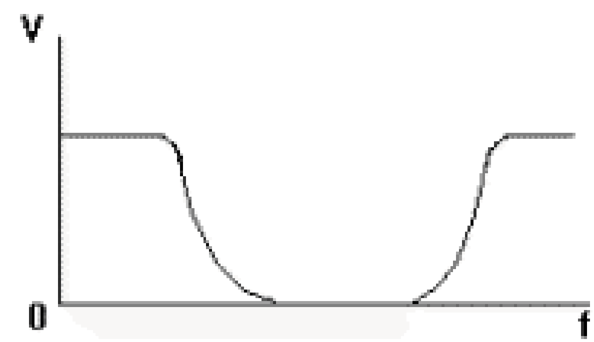
Low Pass



High Pass



Band Pass



Band Stop

Filter Design

[Will consider lowpass; other filters later]

Insights About Filter Design

Filter design in time domain?

- Would be nice to take ideal frequency response and derive h(n) using inverse Fourier transform
  - Elegant & perfect frequency response
  - Any drawbacks?

- Ideal lowpass has impulse response $h(n) = \frac{\sin(\omega_c \pi n)}{\pi n}$
  - Not causal (h(n) nonzero for n<0) → unimplementable
  - Infinite duration → unimplementable
  - Not absolutely summable (recall Gibbs effect) → not BIBO stable

- Conclusion: "ideal" impulse response not so ideal

Filter design by pole/zero placement

- More realistic to design difference equation and carefully place poles and zeros
  - Recall graphical interpretation of LTI system's frequency response
  - Poles near unit circle boost response
  - Zeros near unit circle attenuate

- Guidelines for placing zeros and poles:
  1) Put poles inside unit circle
  - Rationale – ROC will be outside pole with biggest radius → BIBO stable

  2) Poles/zeros will either be real valued or in conjugate pairs
  - Implies real valued difference equation

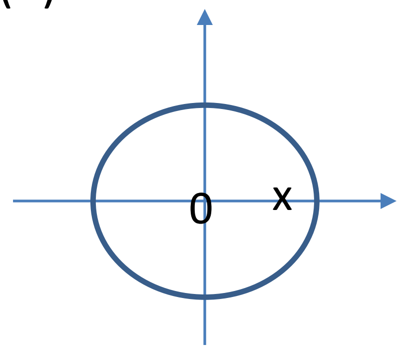
Lowpass Filter Design

[Other filter styles considered later]

Example lowpass

- Illustrating *principles* of filter design through lowpass example
  - Will progress from simple designs to more sophisticated ones over remainder of semester
  - Main idea to place poles near unit circle at lower frequencies; zeros on/near unit circle at higher freqs

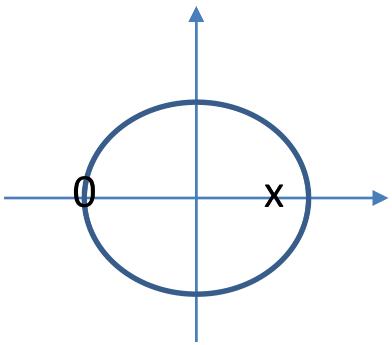
- Consider $H_1(z) = \frac{1-a}{1-az^{-1}} = \frac{(1-a)z}{z-a}$
  - Difference equation implementation, y(n)=ay(n-1)+(1-a)x(n)
  - Pole at p_1=a, zero at z_1=0
  - *Any guesses about 1-a?*



- Frequency response:
  - Peaked at small ω
  - Never zero, because the zero isn't on unit circle (it's at origin)

Better lowpass

- Previous example never had zero response (zero was at origin)
- Let's put zero on unit circle, $H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}} = \frac{1-a}{2}\frac{z+1}{z-a}$
  - More complicated implementation, y(n)=ay(n-1)+(1-a)(x(n)+x(n-1))/2
  - Pole at p_1=a, zero at z_1=-1
  - Zero at z_1=-1 corresponds to frequency $\omega=\pi$
  - *Any guesses about (1-a)/2?*
- Frequency response:
  - Peaked at small ω (as before)
  - Zero at $\omega=\pi$



- More poles/zeros can improve response → more complexity

Filter design example

- Consider $H(z) = \frac{az}{z-b}$
- Want to compute a, b such that
  1) H(ω=0)=1 (perfect lowpass)
  2) |H(ω=π/2)|=1/2 (half response half way to ω=π)

- Solution: H(ω=0)=H(z=1)=a/(1-b)=1 → a=1-b
- |H(ω=π/2)|=|H(z=j)|=|aj/(j-b)|=1/2
- $\left|\frac{aj}{j-b}\right| = \frac{|a|}{\sqrt{1+b^2}} = \frac{|1-b|}{\sqrt{1+b^2}} = \frac{1}{2}$
- Can solve for b, then a

Active learning (Example 5.4.1)

- Let's look at *double pole*, $H(z) = \frac{b_0}{(1-pz^{-1})^2}$
- Want to compute b_0, p such that
  a) $H(\omega=0)=1$
  b) $|H(\omega=\pi/4)|^2=1/2$

a) Convert requirement on $H(\omega=0)$ to z (compute z^* for $\omega=0$)

b) What does $H(z^*)=1$ mean in terms of parameters b_0, p?

Example 5.4.1 continued

- Recall $H(z) = \frac{b_0}{(1-pz^{-1})^2}$ and second condition $|H(\omega=\pi/4)|^2=1/2$

- $H\left(\omega = \frac{\pi}{4}\right) = H(z = e^{j\pi/4}) = \frac{b_0}{(1-pe^{-j\pi/4})^2}$
- Want its magnitude, let's examine denominator
- $\left|1 - pe^{-\frac{j\pi}{4}}\right| = \left|\left[1 - p * cos\left(\frac{\pi}{4}\right)\right] + j\left[\mathrm{p} * \sin\left(\frac{\pi}{4}\right)\right]\right| =$
  $\sqrt{\left[1 - p * cos\left(\frac{\pi}{4}\right)\right]^2 + \left[\mathrm{p} * \sin\left(\frac{\pi}{4}\right)\right]^2} =$
  $\sqrt{1 - 2p * \cos(.) + p^2 cos^2(.) + p^2 sin^2(.)} =$
  $\sqrt{1 + p^2 - 2p * \cos(.)}$

From Lowpass to Other Filters

Lowpass to bandpass

- Lowpass
  - Poles near unit circle at low freqs (to boost output)
  - Zeros near/on unit circle at high freqs

- Bandpass
  - Poles near unit circle at medium freqs → complex conjugate pair
  - Zeros near/on high & low freqs

Example 5.4.2

- Design bandpass with center band at $\omega=\pi/2$, zeros at $\omega_1=0$ and $\omega_2=\pi$, and magnitude response $|H(\omega=4\pi/9)|^2=1/2$
  - Explanation: peak at $\pi/2$ (magnitude 1), decay to half energy at $4\pi/9$

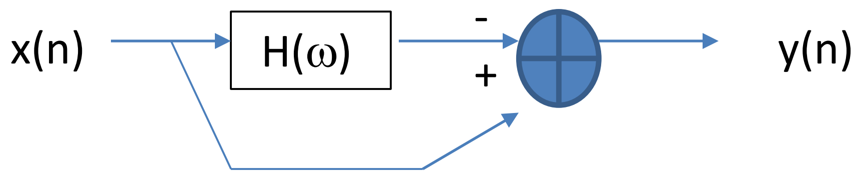
- Solution: select complex conjugate pair of poles, $p_{1,2}=re^{\pm j\pi/2}=\pm jr$
  - r is radius of poles (inside unit circle → r<1)
- Zeros at $z_1=e^{j0}=1$, $z_2=e^{j\pi}=-1$
- $H(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = G\frac{(z-1)(z+1)}{(z-jr)(z+jr)} = G\frac{z^2-1}{z^2+r^2}$
- Still need to determine G (gain) and r (radius)
- Constraint #1: $H\left(\omega=\frac{\pi}{2}\right) = H(z=j) = G\frac{j^2-1}{j^2+r^2} = G\frac{-2}{-1+r^2}$
- We conclude $G=\frac{1-r^2}{2}$; can also incorporate $|H(\omega=4\pi/9)|^2=1/2$

Lowpass to highpass

- Suppose we know (we will soon!) to design good lowpass
- For highpass, let's flip frequency response, $H_{HP}(\omega)= H_{LP}(\omega-\pi)$
- Translating by π radians per sample → time domain modulation by $e^{j\pi n}$
  - $h_{HP}(n)= e^{j\pi n}h_{LP}(n)=(-1)^n h_{LP}(n)$
  - Need to flip signs of odd samples
- Can also show how difference equation for LP can be converted to difference equation for HP

Problem 5.57

- Consider block diagram



- Will want to determine frequency response

a) $H(\omega)$ lowpass with cutoff ω_c, $H(\omega) = \begin{cases} 1, |\omega| < \omega_c \\ 0, \ else \end{cases}$

 – $Y(\omega)=H_{new}(\omega)X(\omega)=-H(\omega)X(\omega)+X(\omega)=[1-H(\omega)]X(\omega)$

 – $H_{new}(\omega)=1-H(\omega)=\begin{cases} 0, |\omega| < \omega_c \\ 1, \ else \end{cases}$ → highpass filter

b) $H(\omega)$ highpass

 – Can see that $H_{new}(\omega)$ lowpass

Example (2013 Midterm; Question 4)

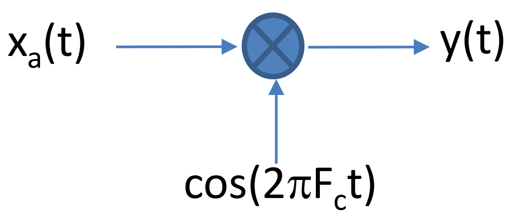
- Derive transfer function for high pass filter with 1 zero & 1 pole:
  - Zero @ $\omega = 0$ on unit circle
  - Pole @ $\omega = \pi$ inside unit circle (want filter to be BIBO stable)
  - Filter should have unit gain @ $\omega = \pi$, i.e., $H(\pi) = 1$), and magnitude response 0.5 at $\omega = \pi/2$, i.e., $H(\pi/2) = 0.5$

Application to AM Demodulation

Problem setting

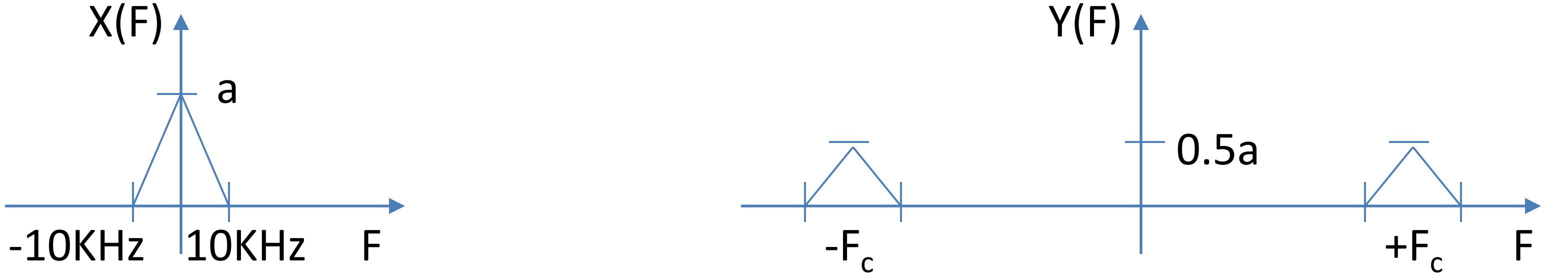
- We've seen lowpass, highpass, …
- Let's see where this can be applied

- Recall our "old friend" amplitude modulation (AM) radio
- Want to demodulate digitally
- Suppose bandwidth is 10 KHz and carrier frequency 900 KHz
- Recall that AM modulates $x_a(t)$ by sinusoid at carrier frequency
  - Suppose F_c=900 KHz



Frequency domain perspective

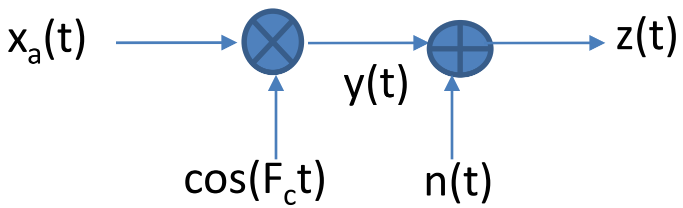
- Recall that cosine corresponds to two complex exponents
- $y(t)=x_a(t)\cos(F_c t) \leftrightarrow Y(F)=0.5X(F+F_c)+0.5X(F-F_c)$



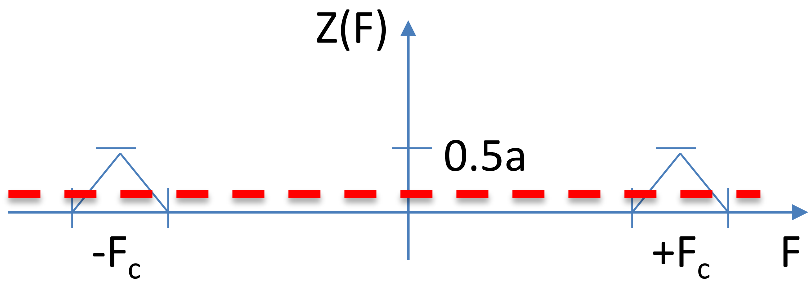
- Two copies of X(F) centered around $\pm F_c$ at half amplitude

Noise

- Real world system also contains noise



- Receiver sees z(t)=y(t)+n(t), n(t) noise (suppose it's wideband)



- Want to recover $x_a(t)$

Analog recovery of input?

- Would be nice to block noise with bandpass filter

- Can we do so in analog?
  - Hard to design quality analog filters
  - Carrier frequency way bigger than bandwidth of input

- Let’s try to do so digitally

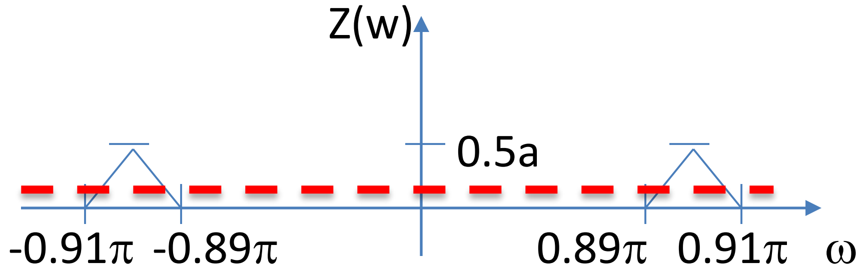
Naïve digital approach

- Suppose we sample noisy z(t) and apply bandpass digitally
- Nyquist rate 2*(900KHz+10KHz)=1.82M samples/sec

- Might be complicated to construct clock at exactly 1.82M
- Use standard A/D at 2M samples/sec

Spectrum of sampled signal

- Let’s evaluate Fourier of sampled digital signal
- π radians/sample correspond to 1MHz
- 0.91M corresponds to 0.91π



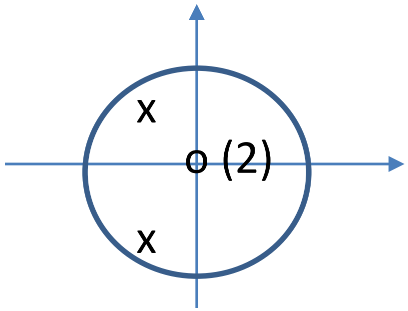
- Will use digital bandpass between 0.89π and 0.91π
  - There will be some noise in passband
  - Will multiply signal by cos(0.9πn) to bring to DC

- *Will revisit this with better digital design soon*

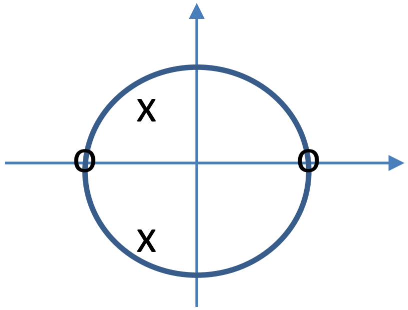
More Filters

Digital resonators

- How should we design our digital bandpass filter (for AM)?
- Simple approach is two poles at passband frequencies ($\pm 0.9\pi$)
- Where will zeros be?
  - Both at origin → all frequencies (unit circle) influenced the same
  - Frequency response never zero

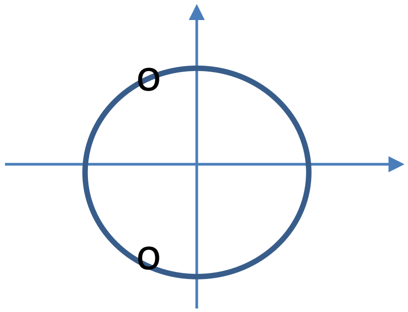


  - At $z=\pm 1$ ($\omega \in \{0,\pi\}$) → nice attenuation of frequencies away from poles

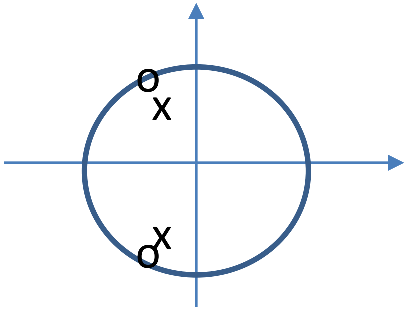


Notch filters

- Notch filter has deep notches that reject some frequencies
- We place zeros on unit circle
- Useful for rejecting very specific frequencies
  - Example: power lines at 60 Hz often seep into signal → remove w/notch



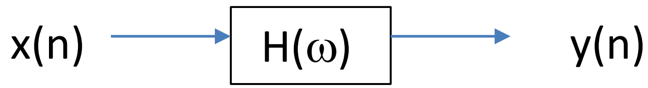
- Challenge: zeros also reject band nearby
- Solution: add poles near zeros
  - Zero/pole cancels out as we move away from notch frequency



Deconvolution

Motivation

- Consider "standard" LTI system



- We are often given y(n) and asked to *reconstruct* x
  - Example: x(n) is speech signal, H is room (echos), want to clean up x
- Also called *deconvolution* or *inverse problem*

- Naïve solution: apply $G=H^{-1}$

Challenges

- If H unknown, can't compute H^{-1}
  - Can estimate H by transmitting *training signal*
  - In theory, need only do so once
  - In practice (communication systems), H often changes (things move near the mobile phone → signals bounce around differently → different H)

- Messy even if H perfectly known
  - If z=y+n, applying $G=H^{-1}$ might amplify noise (for small $H(\omega)$)

- Also need to make G causal & stable

- Complicated area