ECE 421
Introduction to Signal Processing

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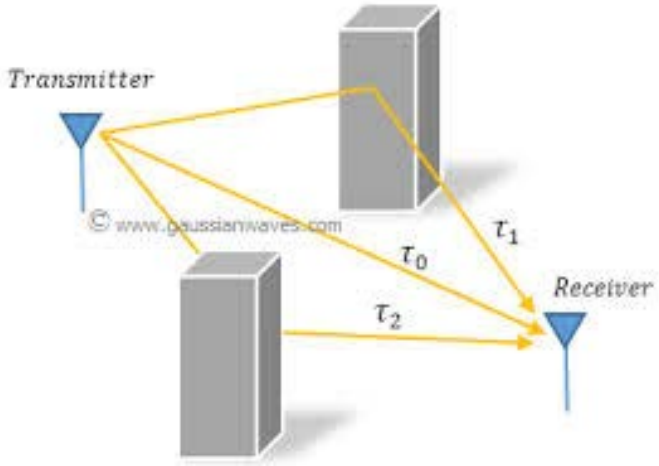
Fast Fourier Transform

[Reading material: Chapter 8

Motivation

- We've discussed that convolution is very useful (filtering)

- Recall multipath example
  - 100M samples/sec
  - 1000 taps
  - 100B multiplications/sec
  - 100B additions/sec



- Filtering can be performed as y=IDFT(DFT(x)DFT(h))
  - DFT(h) can be pre-computed (once)
  - Computing IDFT & DFT analogous (minus signs in exponents)

- *How fast can we compute DFT?*

Naïve approach

- Recall $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Matlab:

```
N=2000; % signal length
x=randn(N,1); % random input
xf=zeros(N,1); % initialize DFT coeffs
tic % start clock
for k=0:N-1 % loop over k
  xf(k+1)=exp(-j*2*pi*k*(0:N-1)/N)*x; % computes everything one line
end;
toc
```

- Complicated line: inner product between row exp(…) column x

Why is it slow?

- Main line runs N times
  - N complex multiplications
  - N-1 complex additions
  - N complex exponentials
  - More real-valued arithmetic (set up exponents)

- *Quadratic* computation

What's wrong with quadratic runtime?

- Let's compare N^2 (DFT runtime) with $N \times \log_2(N)$ for FFT

N	N^2	$N \log_2(N)$
10^3	10^6 (feasible 1960s)	~10^4
10^6	10^{12} (feasible 2000s)	~2×10^7
10^9	10^{18} **infeasible**	~3×10^{10}

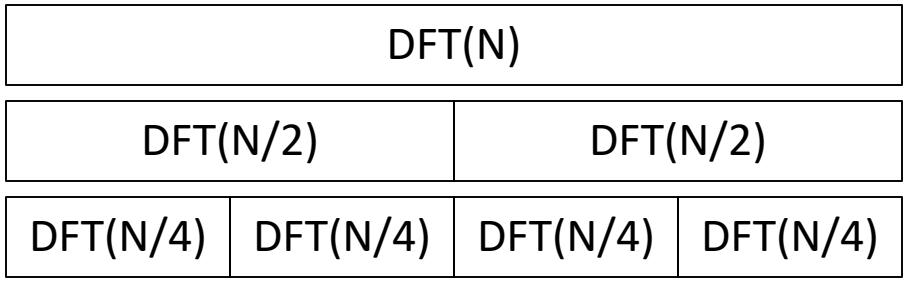
- Reminder about computer speeds:
  - 1960s: dozens to thousands operations/second
  - 1980s: millions
  - 2000s: billions
  - Modern general purpose graphics processing units: 100x

Divide and Conquer

How can we improve DFT speed?

- *Divide* into blocks
- Compute DFT of each block (*conquer*)
- Merge - “book keeping” to patch together DFT’s of blocks
  - Merging step needs to be fast (linear in N)

- Trick: use divide & conquer hierarchically within blocks



Example: Sorting

- Let's see another (simpler) $N \times \log_2(N)$ algorithm

- Want to *sort* set of numbers, x={1,9, -2, 3, 6, -1, 7, 4}
  - Step1: partition into x_1={1,9, -2, 3} and x_2={6, -1, 7, 4}
  - Step2: sort each component
    - Sort(x_1)={-2, 1, 3, 9}
    - Sort(x_2)={-1,4,6,7}
  - Merging step:
    - -2 < -1 → Sort(x)←{-2}, Sort(x_1)←{1, 3, 9}
    - -1 < 1 → Sort(x)←{-2, -1}, Sort(x_2)←{4, 6, 7}
    - 1 < 4 → Sort(x)←{-2, -1, 1}, Sort(x_1)←{3, 9}
    - …
- Can see that merging requires runtime linear in N
- "Mergesort" requires $N \times \log_2(N)$ runtime

How fast is divide & conquer?

- Denote runtime of size-n problem by t(n)
- Suppose merge(N) takes C_1N time

- Result: $t(N) \leq C_2 N \cdot \log_2(N)$
- Proof: (induction on N)
  - Basis case: take N=2 or 4…
  - Inductive step: we assume $t(N) \leq C_2 N \cdot \log_2(N)$
  - Want to show $t(2N) \leq C_2 2N \cdot \log_2(2N)$
  - $t(2N) \leq 2t(N) + C_1 N \leq 2C_2 N \log_2(N) + C_1 N = C_2 N[2\log_2(N) + \frac{C_1}{C_2}]$
  - Suppose $C_2 \geq C_1$ (else increase C_2)
  - $t(2N) \leq C_2 N[2\log_2(N) + 1] < C_2 N[2\log_2(N) + 2]$
    $= 2C_2 N[\log_2(N) + 1] = 2C_2 N \log_2(2N)$

DFT Using Divide and Conquer

How to apply divide & conquer

- Partition N into L*M
- Compute DFT for L blocks, each of length M
  - Can compute hierarchically by partitioning M further
- Merge L blocks

- Runtime will be proportional to $N \cdot \log_2(N)$

- If N doesn't factor nicely into prime numbers, can zero-pad

- Example Matlab implementations on course webpage

How to “merge” DFT’s in linear time?

- *Let’s relate length-N DFT to L length-M DFTs (linear time)*
- Take N=L*M, n=l+mL, k=Mp+q
- $X(p,q) = X(k = Mp + q) = \sum_{n=0}^{N-1} x(n) w_N^{kn} =$
  $\sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(n = l + mL) w_N^{(Mp+q)(l+mL)} =$
  $\sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m) w_N^{(Mp+q)(l+mL)}$
- Let’s rewrite w term:

  $w_N^{(Mp+q)(l+mL)} = w_N^{Mpl+ql+MpmL+qmL}$

  $= w_N^{Mpl} w_N^{ql} w_N^{MpmL} w_N^{qmL}$

  $= w_L^{pl} w_N^{ql} w_M^{qm}$
- Note that $w_N^M = wL, w_N^L = wM, w_N^{ML} = 1$

How to “merge” DFT’s continued

- We now have:

$$X(p,q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m) w_N^{(Mp+q)(l+mL)}$$

$$= \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m) w_L^{pl} w_N^{ql} w_M^{qm}$$

$$= \sum_{l=0}^{L-1} w_L^{pl} \left\{ w_N^{ql} \sum_{m=0}^{M-1} x(l,m)\, w_M^{qm} \right\}$$

$$= \sum_{l=0}^{L-1} w_L^{pl} \left\{ w_N^{ql} DFT_M(x(l,\cdot)) \right\}$$

- Conclusion:

1) Compute L DFT’s, each of M points
2) Modulate by w_N^{ql} (this is new – required for merge)
3) Compute M DFT’s, each of L points (made fast w/small L)