Bases and LTI systems

Dror Baron

This supplement provides more details about bases and linear time invariant (LTI) systems.

Orthonormal bases: We discussed in class that if $[v_1, \ldots, V_N]$ is an orthonormal basis that spans a vector space V, then a vector $x \in V$ can be expressed as

$$x = \sum_{i=1}^{N} \alpha_i v_i,$$

where the coefficients, $(\alpha_i)_{i=1}^N$, satisfy

$$\alpha_i = \langle x, v_i \rangle.$$

Other bases: But what if $[v_2, \ldots, v_N]$ is not orthonormal? We will now show that the above results are incorrect.

To see this, we provide several simple counter examples. Begin with a very simple example where $V = \mathbb{R}$. That is, our vector space is the one-dimensional space of real numbers. Consider a spanning basis, $v_1 = 2$. Now choose some vector $x \in V$ that takes value $x = \beta$. To represent x using v_1, the coefficient α_1 must be $\frac{\beta}{2}$, which provides the desired result,

$$\alpha_1 v_1 = \frac{\beta}{2} \cdot 2 = \beta = x.$$

Note, however, that

$$\langle x, v_1 \rangle = \beta \cdot 2 = 2\beta,$$

which differs from $\alpha_1 = \frac{\beta}{2}$.

Our second example is somewhat more complicated, and $V = \mathbb{R}^2$ is a two-dimensional vector space. Suppose that our basis takes on the following values,

$$[v_1 \ v_2] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let's compute the inner products between basis vectors, $\langle v_1, v_1 \rangle = 2 \cdot 2 + 0 \cdot 0 = 4$, $\langle v_1, v_2 \rangle = 2 \cdot 0 + 0 \cdot 1 = 0$, $\langle v_2, v_1 \rangle = 0 \cdot 2 + 1 \cdot 0 = 0$, and $\langle v_2, v_2 \rangle = 0 \cdot 0 + 1 \cdot 1 = 1$. Consider $x = [\beta \ 0]^T$, where $[\cdot]^T$ denotes transpose. It is easy to see that $v = \frac{\beta}{2} \cdot v_1 + 0 \cdot v_2$, meaning that $\alpha_1 = \frac{\beta}{2}$ and $\alpha_2 = 0$. It can be shown that when the cross terms are zero, which implies that the basis is orthogonal (but not necessarily orthonormal, which further requires all basis vectors to have unit norm), then

$$\alpha_i = \frac{1}{\langle v_i, v_i \rangle} \langle x, v_i \rangle, \tag{1}$$

where the correction term, $\frac{1}{\langle v_i, v_i \rangle} = \frac{1}{\|v_i\|^2}$, ensures proper normalization.

So far our examples have considered orthogonal bases. Consider now the following non-orthogonal basis,

$$[v_1 \ v_2] = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

It can easily be shown that the two vectors within this basis each have unit norm. However, if we use our modified expression (1) for computing the coefficients corresponding to the same vector $x = [\beta \ 0]^T$ from the second example, then we obtain

$$\alpha_1 = \frac{1}{\langle v_1, v_1 \rangle} \langle x, v_1 \rangle = \frac{1}{1^2}(1 \cdot \beta + 0 \cdot 0) = \beta$$

and

$$\alpha_2 = \frac{1}{\langle v_2, v_2 \rangle} \langle x, v_2 \rangle = \frac{1}{\frac{1}{2} + \frac{1}{2}} \left(\frac{1}{\sqrt{2}} \cdot \beta + \frac{1}{\sqrt{2}} \cdot 0 \right) = \frac{\beta}{\sqrt{2}}.$$

We can see that the formula (1) fails due to v_1 and v_2 not being orthogonal.

Linear time invariant systems: What is a linear time invariant (LTI) system? We say that a system T that maps inputs x to outputs y is *linear* is

$$T(\alpha x_a + \beta x_b) = \alpha T(x_a) + \beta T(x_b)$$

for all scalar constants α, β and all inputs x_a and x_b. For $V = \mathbb{R}$, an example of a linear T is $T_1(x) = 2x$. Similarly, $T_2(x) = 2x + 2$ is non-linear, because we can choose $x = 0$ and $\alpha = 2$ such that $T_2(x) = T_2(0) = 2 \cdot 0 + 2 = 2$, $\alpha T_2(x) = 2 \cdot 2 = 4$, but on the other hand $T_2(\alpha x) = T_2(2 \cdot 0) = T_2(0) = 2 \cdot 0 + 2 = 2$. (Note that our example considers a one dimensional vector space V to keep things simple, while LTI systems typically deal with sequences of numbers at the input and output.)

We say that T is *time invariant* if the time shift operator, denoted by $\sigma(\cdot)$, has the same effect whether applied before or after T, i.e.,

$$\sigma(T(x)) = T(\sigma(x)).$$

An example of a time invariant system is $y = T_3(x) = \sigma(x)$. That is, $y(n)$ is a delayed version of $x(n)$, i.e., $y(n) = x(n-1)$. In this case, $y(n) = \sigma(T(x)) = x(n-2) = T(x(n-1)) = T(\sigma(x))$. And an example of a system that is time variant (not time invariant) is $y(n) = T_4(x) = n \cdot x(n)$. It can be shown that modulation by the time index n ruins the time variance property.

A system that is both linear and time invariant is called linear time invariant (LTI). LTI systems have the fortunate property that they can be expressed as *convolution*,

$$y(n) = \sum_{i=-\infty}^{+\infty} h_i x_{n-i},$$

where the sequence h is called the convolution kernel. A convenient property is that convolution in the time domain becomes multiplication in the Fourier domain, $Y(\omega) = H(\omega)X(\omega), \forall\omega$.