Simple Single Letter Bound

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This supplement provides more details about a simple single letter bound derived by the instructor and coauthors in 2006.

Single letter expressions: In information theory, a single letter expression describes the limiting best-possible performance of a communication system in the limit of large block lengths. One example of a single letter expression is the channel capaxity C, which is the largest number of bits that can be communicated reliably over a noisy channel, where we normalize by the number of channel uses. In lossy data compression, $R(D)$ is the rate of bits required to describe an input source up to average distortion D. The notion of rate is that we normalize the number of bits used to compress an input by the length of the input.

Bounds vs. performance limits: A single letter performance limit characterizes the best-possible performance. It is possible to get very close to this limit (this is known as the *achievable* part of the performance limit). Similarly, it is impossible to do better than the limit (known as a *converse* result).

Many information theoretic problems are complicated, and a single letter performance limit, which is comprised of matching achievable and converse bounds, is not available. Instead, there are often achievable and converse *bounds*. These bounds are different. That is, the best known achievable performance is not as good as the performance ruled out by the converse.

Single letter bounds in compressed sensing: Around 2006–2009, many authors provided various single letter bounds in compressed sensing (CS). One of these results was by the instructor and his coauthors, Shriram Sarvotham and Richard Baraniuk, in 2006. This result involved looking at CS as a communication system. The matrix vector product $y = \sqrt{\gamma}\Phi x$ can be interpreted as an input x going through a source encoder (compression algorithm) followed by a channel encoder. Next, the channel adds noise z to y_0, resulting in $y = y_0 + z$. Finally, the recovery algorithm of CS corresponds to a channel decoder followed by a source decoder (a decompressor).

The following discussion will be an informal presentation of the ideas in our single letter bound from 2006. Suppose that we communicate M measurements, where M is large (later we will normalize various sizes, resulting in a single letter type of bound). Due to the capacity C of the channel, it is impossible to communicate more than $\frac{M}{C}$ reliable bits of information over the channel (again, this is rather informal; details appear in the paper). Recall that our input is of length N, and if we encode at rate $R(D)$, then the number of bits at the output of the source encoder is roughly $NR(D)$. Clearly, the number of bits that can be communicated reliably over the channel must be no smaller than the number of bits at the output of the source encoder,

$$NR(D) \leq \frac{M}{C}.$$

Therefore, we get a bound on the measurement rate $\delta = \frac{M}{N}$,

$$\delta = \frac{M}{N} \geq \frac{R(D)}{C}.$$

Our single letter bound is imprecise; it is not a performance limit. However, it still provides plenty of intuition about CS systems. For example, we can consider mean square error, and allow for the distortion metric D in the rate distortion (RD) function to relate the rate R to mean square error. Alternately, we could be interested in support set detection or some other sort of error in the system, which corresponds to a different RD function. Additionally, the bound allows for different types of channels. Gaussian channels, Poisson channels, and more.