

# Signal Estimation with Additive Error Metrics in Compressed Sensing

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## Abstract

Compressed sensing typically deals with the estimation of a system input from its noise-corrupted linear measurements, where the number of measurements is smaller than the number of input components. The performance of the estimation process is usually quantified by some standard error metric such as squared error or support set error. In this correspondence, we consider a noisy compressed sensing problem with any additive error metric. Under the assumption that the relaxed belief propagation method matches Tanaka's fixed point equation, we propose a general algorithm that estimates the original signal by minimizing the additive error metric defined by the user. The algorithm is a pointwise estimation process, and thus simple and fast. We verify that our algorithm is asymptotically optimal, and we describe a general method to compute the fundamental information-theoretic performance limit for any additive error metric. We provide several example metrics, and give the theoretical performance limits for these cases. Experimental results show that our algorithm outperforms methods such as relaxed belief propagation (relaxed BP) and compressive sampling matching pursuit (CoSaMP), and reaches the suggested theoretical limits for our example metrics.

## Index Terms

Belief propagation, compressed sensing, error metric, estimation theory.

## I. INTRODUCTION

### A. Motivation

Compressed sensing [2–4] has received a great deal of attention in recent years, because it deals with signal reconstruction problems with far fewer samples than required by the Nyquist rate, greatly reducing the sampling rates required in signal processing applications, such as cameras, medical scanners, and high speed radar [5].

Consider a linear system,

$$\mathbf{w} = \Phi \mathbf{x}, \quad (1)$$

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where the system input  $\mathbf{x} \in \mathbb{R}^N$  is independent and identically distributed (i.i.d.), and the random linear mixing matrix  $\Phi \in \mathbb{R}^{M \times N}$  is sparse and known (typically  $M < N$ ). The vector  $\mathbf{w} \in \mathbb{R}^M$  is called the measurement of  $\mathbf{x}$ , and is passed through a bank of separable channels characterized by conditional distributions,

$$f_{\mathbf{Y}|\mathbf{W}}(\mathbf{y}|\mathbf{w}) = \prod_{i=1}^M f_{Y_i|W}(y_i|w_i), \quad (2)$$

where  $\mathbf{y}$  is the channel output vector, and  $(\cdot)_i$  denotes the  $i$ th element of a vector. Note that the channels are general and are not restricted to Gaussian. We observe the channel output  $\mathbf{y}$ , and want to estimate the original input signal  $\mathbf{x}$  from  $\mathbf{y}$  and  $\Phi$ . The remainder of this correspondence follows this system modeled by (1) and (2) (see Rangan [6] for detailed assumptions about the system model).

The performance of the estimation process is often characterized by some error metric that quantifies the distance between the estimated and the original signals. For a signal  $\mathbf{x}$  and its estimate  $\hat{\mathbf{x}}$ , both of length  $N$ , the error between them is the summation over the component-wise errors,

$$D(\hat{\mathbf{x}}, \mathbf{x}) = \sum_{j=1}^N d(\hat{x}_j, x_j). \quad (3)$$

For example, if the metric is absolute error, then  $d(\hat{x}_j, x_j) = |\hat{x}_j - x_j|$ ; for squared error,  $d(\hat{x}_j, x_j) = (\hat{x}_j - x_j)^2$ .

Squared error is one of the most popular error metrics in various problems, due to many of its mathematical advantages. For example, minimum mean squared error (MMSE) estimation provides both variance and bias information about an estimator [7], and in the Gaussian case it is linear and thus often easy to implement [8]. However, there are applications where MMSE estimation is inappropriate, for example because it is sensitive to outliers [9, 10]. Therefore, alternative error metrics, such as mean absolute error (median), mean cubic error, or Hamming distance, are used instead. Considering the significance of various types of error metrics other than squared error, a general estimation algorithm that can minimize any desired error metric is of interest.

## B. Related work

As mentioned above, squared error is most commonly used as the error metric in estimation problems given by (1) and (2). Mean-squared optimal analysis and algorithms were introduced in [11–15] to estimate a signal from measurements corrupted by Gaussian noise; in [6, 16, 17], further discussions were made about the circumstances where the output channel is arbitrary, while, again, the MMSE estimator was put forth. Another line of work, based on a greedy algorithm called *orthogonal matching pursuit*, was presented in [18, 19] where the mean squared error decreases over iterations. Absolute error is also under intense study in signal estimation. For example, an efficient sparse recovery scheme that minimizes the absolute error was provided in [20, 21]; in [22], a fundamental analysis was offered on the minimal number of measurements required while keeping the estimation error within a certain range, and absolute error was one of the metrics concerned. Support recovery error is another metric of great importance, for example because it relates to properties of the measurement matrices [23]. The authors of [23–25] discussed the support error rate when recovering a sparse signal from its noisy measurements; support-related performance metrics were applied in the derivations of theoretical limits on the sampling rate for signal

recovery [26, 27]. The readers may notice that previous work only paid attention to limited types of error metrics. What if absolute error, cubic error, or other non-standard metrics are required in a certain application?

### C. Contributions

In this correspondence: (i) we suggest a pointwise Bayesian estimation algorithm that minimizes an arbitrary additive error metric; (ii) we prove that the algorithm is optimal; (iii) we study the fundamental information-theoretic performance limit of an estimation for a given metric; (iv) we derive the performance limits for minimum mean absolute error, minimum mean support error, and minimum mean weighted-support error estimators, and obtain the receiver operating characteristic (ROC) of the modeled system by weighted-support error. This algorithm is based on the assumption that the relaxed belief propagation (BP) method [6] converges to a set of degraded scalar Gaussian channels [11, 13, 14, 16]. The relaxed BP method is well-known to be optimal for the squared error, while we further show that the relaxed BP method can do more – because the sufficient statistics are given, other additive error metric can also be minimized with one more simple and fast step. This is convenient for users who desire to recover the original signal with a non-standard additive error metric. Simulation results show that our algorithm outperforms algorithms such as relaxed BP [6], which is optimal for squared error, and compressive sampling matching pursuit (CoSaMP) [19], a greedy reconstruction algorithm. Moreover, we compare our algorithm with the suggested theoretical limits for minimum mean absolute error (MMAE), minimum mean support error (MMSuE), and minimum mean weighted-support error (MMWSE), and illustrate that our algorithm is optimal.

The remainder of the correspondence is arranged as follows: we review relaxed BP in Section II, and then describe our estimation algorithm and discuss its performance in Section III. Simulation results are given in Section IV, while conclusions appear in Section V. Some mathematical details appear in appendices.

## II. REVIEW OF RELAXED BELIEF PROPAGATION

Before describing the estimation algorithm, a review of the relaxed BP method [6, 12] is helpful.

Belief Propagation (BP) [28] is an iterative method used to compute the marginals of a Bayesian network. Consider the bipartite graph, called a *Tanner* or *factor* graph, shown in Figure 1, where circles represent random variables (called *variable nodes*), and related variables are connected through functions (represented by squares, called *factor nodes* or *function nodes*) that indicate their dependence. In standard BP, there are two types of messages passed through the nodes: messages from variable nodes to factor nodes,  $m_{x \rightarrow y}$ , and messages from factor nodes to variable nodes,  $m_{y \rightarrow x}$ . If we denote the set of function nodes connected to the variable  $x$  by  $N(x)$ , the set of variable nodes connected to the function  $y$  by  $N(y)$ , and the factor function at node  $y$  by  $\Psi_y$ , then the two types of messages are defined as follows [28]:

$$\begin{aligned} m_{x \rightarrow y} &= \prod_{k \in N(x) \setminus y} m_{k \rightarrow x}, \\ m_{y \rightarrow x} &= \sum_{\ell \in N(y) \setminus x} \Psi_y m_{\ell \rightarrow y}. \end{aligned}$$

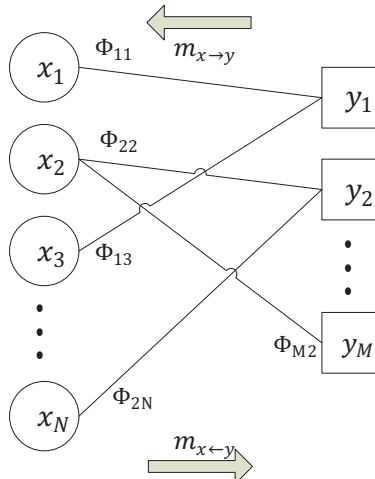


Fig. 1: Tanner graph for relaxed belief propagation.

Inspired by the basic BP idea described above, the authors of [12, 16] developed iterative algorithms for estimation problems in linear mixing systems. In the Tanner graph, an input vector  $x = [x_1, x_2, \dots, x_N]$  is associated with the variable nodes (input nodes), and the output vector  $y = [y_1, y_2, \dots, y_M]$  is associated with the function nodes (output nodes). If  $\Phi_{ij} \neq 0$ , then nodes  $x_j$  and  $y_i$  are connected to an edge  $(i, j)$ , where the set of such edges  $E$  is defined as  $E = \{(i, j) : \Phi_{ij} \neq 0\}$ .

In standard BP methods [29–31], the distribution functions of  $x_j$  and  $w_i$  as well as the channel distribution function  $f_{Y|W}(y_i|w_i)$  were set to be the messages passed along the graph, but it is difficult to compute those distributions, making the standard BP method computationally expensive. In [12], a simplified algorithm, called *relaxed belief propagation*, was suggested. In this algorithm, means and variances replace the distribution functions themselves and serve as the messages passed through nodes in the Tanner graph, greatly reducing the computational complexity. In [6, 16, 17, 32], this method was extended to a more general case where the channel is not necessarily Gaussian.

In Rangan [6], the relaxed BP algorithm generates two sequences,  $q_j(t)$  and  $\mu_j(t)$ , where  $t \in \mathbb{Z}^+$  denotes the iteration number. Under the assumptions that the signal dimension  $N \rightarrow \infty$ , the iteration number  $t \rightarrow \infty$ , and the ratio  $M/N$  is fixed, the sequences  $q_j(t)$  and  $\mu_j(t)$  converge to sufficient statistics for the linear mixing channel observation  $\mathbf{y}$  (2). More specifically, in the large system limit, the conditional distribution  $f(x_j|q_j(t), \mu_j(t))$  converges to the conditional distribution  $f(x_j|\mathbf{y})$ , where  $q_j(t)$  can be regarded as a Gaussian-noise-corrupted version of  $x_j$ , and  $\mu_j(t)$  is the noise variance,

$$q_j(t) = x_j + v_j, \quad (4)$$

where  $v_j \sim \mathcal{N}(0, \mu_j(t))$  for  $j = 1, 2, \dots, N$ . It has been shown [6] that  $\mu_j(t)$  converges to a fixed point that satisfies Tanaka's equation, which has been analyzed in detail (cf. [11, 13, 14, 17, 30, 33, 34]). We define the limits

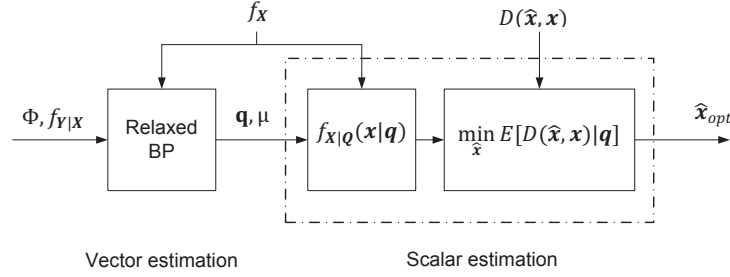


Fig. 2: The structure of the metric-optimal estimation algorithm.

of the two sequences,

$$\begin{aligned} \lim_{t \rightarrow \infty} q_j(t) &= q_j, \\ \lim_{t \rightarrow \infty} \mu_j(t) &= \mu, \end{aligned}$$

for  $j = 1, 2, \dots, N$ . Note that all the scalar Gaussian channels (4) have the same noise variance  $\mu$ . We now simplify equation (4) as follows,

$$q_j = x_j + v_j, \quad (5)$$

where  $v_j \sim \mathcal{N}(0, \mu)$  for  $j = 1, 2, \dots, N$ .

### III. ESTIMATION ALGORITHM

#### A. Algorithm

The structure of our metric-optimal algorithm is illustrated in the dashed box in Figure 2. The inputs of the algorithm are: (i) a distribution function  $f_{\mathbf{x}}(\mathbf{x})$ , the prior of the original input  $\mathbf{x}$ ; (ii) a vector  $\mathbf{q} = (q_1, q_2, \dots, q_N)$ , the outputs of the scalar Gaussian channels computed by relaxed BP [6]; (iii) a scalar  $\mu$ , the variance of the Gaussian noise in (5); and (iv) an error metric function  $D(\hat{\mathbf{x}}, \mathbf{x})$  specified by the user. The vector  $\mathbf{q}$  and the scalar  $\mu$  are the outputs of the relaxed BP method by Rangan [6], and in particular we generate  $\mathbf{q}$  and  $\mu$  using the software package ‘‘GAMP’’ [35].

Because the scalar channels have additive Gaussian noise, and that the variances of the noise are all  $\mu$ , we can compute the conditional probability density function  $f_{\mathbf{x}|\mathbf{q}}(\mathbf{x}|\mathbf{q})$  from Bayes’ rule:

$$\begin{aligned} f_{\mathbf{x}|\mathbf{q}}(\mathbf{x}|\mathbf{q}) &= \frac{f_{\mathbf{q}|\mathbf{x}}(\mathbf{q}|\mathbf{x})f_{\mathbf{x}}(\mathbf{x})}{f_{\mathbf{q}}(\mathbf{q})} \\ &= \frac{f_{\mathbf{q}|\mathbf{x}}(\mathbf{q}|\mathbf{x})f_{\mathbf{x}}(\mathbf{x})}{\int f_{\mathbf{q}|\mathbf{x}}(\mathbf{q}|\mathbf{x})f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}}, \end{aligned} \quad (6)$$

where

$$f_{\mathbf{q}|\mathbf{x}}(\mathbf{q}|\mathbf{x}) = \frac{1}{\sqrt{(2\pi\mu)^N}} \exp\left(-\frac{\|\mathbf{q} - \mathbf{x}\|_2^2}{2\mu}\right).$$

Given an error metric  $D(\hat{\mathbf{x}}, \mathbf{x})$ , the optimal estimand  $\hat{\mathbf{x}}_{opt}$  is generated by minimizing the conditional expectation of the error metric  $E[D(\hat{\mathbf{x}}, \mathbf{x})|\mathbf{q}]$ , which is easy to compute using  $f_{\mathbf{x}|\mathbf{q}}(\mathbf{x}|\mathbf{q})$ :

$$E[D(\hat{\mathbf{x}}, \mathbf{x})|\mathbf{q}] = \int D(\hat{\mathbf{x}}, \mathbf{x})f_{\mathbf{x}|\mathbf{q}}(\mathbf{x}|\mathbf{q})d\mathbf{x}.$$

Then,

$$\begin{aligned}\hat{\mathbf{x}}_{\text{opt}} &= \arg \min_{\hat{\mathbf{x}}} E[D(\hat{\mathbf{x}}, \mathbf{x})|\mathbf{q}] \\ &= \arg \min_{\hat{\mathbf{x}}} \int D(\hat{\mathbf{x}}, \mathbf{x}) f_{\mathbf{x}|\mathbf{Q}}(\mathbf{x}|\mathbf{q}) d\mathbf{x}.\end{aligned}\quad (7)$$

The conditional probability  $f_{\mathbf{x}|\mathbf{Q}}(\mathbf{x}|\mathbf{q})$  is separable, because the parallel scalar Gaussian channels (5) are separable and  $f_{\mathbf{x}}(\mathbf{x})$  is i.i.d. Moreover, the error metric function  $D(\hat{\mathbf{x}}, \mathbf{x})$  (3) is also separable. Therefore, the problem reduces to scalar estimation [8],

$$\begin{aligned}\hat{x}_{\text{opt},j} &= \arg \min_{\hat{x}_j} E[d(\hat{x}_j, x_j)|q_j] \\ &= \arg \min_{\hat{x}_j} \int d(\hat{x}_j, x_j) f_{x_j|q_j}(x_j|q_j) dx_j,\end{aligned}\quad (8)$$

for  $j = 1, 2, \dots, N$ . Equation (8) minimizes a single-variable function. In Section III-C, we show how to perform this minimization in three example cases.

### B. Theoretical results

Having discussed the algorithm, we now give a theoretical justification for its performance.

*Claim 1: Given the system model described by (1), (2) and an error metric  $D(\hat{\mathbf{x}}, \mathbf{x})$  of the form defined by (3), as the signal dimension  $N \rightarrow \infty$  and the measurement ratio  $M/N$  is fixed, the optimal estimand of the input signal is given by*

$$\hat{\mathbf{x}}_{\text{opt}} = \arg \min_{\hat{\mathbf{x}}} E [D(\hat{\mathbf{x}}, \mathbf{x})|\mathbf{q}],$$

where the vector entries  $\mathbf{q} = (q_1, q_2, \dots, q_N)$  are the outputs of the scalar Gaussian channels (5).

The rationale for Claim 1 is as follows. Because the probability density function  $f_{X_j|\mathbf{Y}}(x_j|\mathbf{y})$  is statistically equivalent to  $f_{X_j|Q_j}(x_j|q_j)$  in the large system limit, once we know the value of  $\mu$ , estimating each  $x_j$  from all channel outputs  $\mathbf{y} = (y_1, y_2, \dots, y_M)$  is equivalent to estimating  $x_j$  from the corresponding scalar channel output  $q_j$ . The relaxed BP method [6] calculates the sufficient statistics  $q_j$  and  $\mu$ . Therefore, an estimator based on minimizing the conditional expectation of the error metric,  $E(D(\hat{\mathbf{x}}, \mathbf{x})|\mathbf{q})$ , gives an asymptotically optimal result.

Claim 1 states that, in the large system limit, the estimator satisfying (7) is optimal, because it minimizes the conditional expectation of the error metric. The key point in the estimation problem is to obtain the posterior  $f_{X_j|\mathbf{Y}}$ . Fortunately, the relaxed BP algorithm provides an asymptotically optimal method to decouple the mixing channels and thus an equivalent posterior  $f_{X_j|Q_j}(x_j|q_j)$  can be computed easily, and our algorithm utilizes this convenient feature.

Following Claim 1, we can compute the minimum expected error achievable by *any* estimation algorithm for any additive error metric  $D(\hat{\mathbf{x}}_{\text{opt}}, \mathbf{x})$ . This minimum expected error is the fundamental information-theoretic performance limit of interest in this problem; no estimation algorithm can out-perform this limit. At the same time, we will see in Section III-C that, for three example error metrics, our BP-based algorithm *matches* the performance of the information-theoretic limit, and is thus optimal.

*Claim 2:* With the optimal estimand  $\hat{\mathbf{x}}_{opt}$  determined by (7), the minimum mean user-defined error (MMUE) is given by

$$MMUE(f_{\mathbf{X}}, \mu) = \int_{R(\mathbf{Q})} \left( \int_{R(\mathbf{X})} D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) \left( \frac{1}{\sqrt{(2\pi\mu)^N}} \exp\left(-\frac{\|\mathbf{q} - \mathbf{x}\|^2}{2\mu}\right) \right) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \right) d\mathbf{q}, \quad (9)$$

where  $R(\cdot)$  represents the range of a variable, and  $\mu$  is the variance of the noise of the scalar Gaussian channel (5).

Equation (9) can be derived in the following steps.

$$\begin{aligned} MMUE(f_{\mathbf{X}}, \mu) &= E[D(\hat{\mathbf{x}}_{opt}, \mathbf{x})] \\ &= \int_{R(\mathbf{Q})} E_{\mathbf{Q}} \left[ E[D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) | \mathbf{q}] \right] f_{\mathbf{Q}}(\mathbf{q}) d\mathbf{q} \\ &= \int_{R(\mathbf{Q})} E[D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) | \mathbf{q}] f_{\mathbf{Q}}(\mathbf{q}) d\mathbf{q} \\ &= \int_{R(\mathbf{Q})} \left( \int_{R(\mathbf{X})} D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) f_{\mathbf{X}|\mathbf{Q}}(\mathbf{x}|\mathbf{q}) d\mathbf{x} \right) f_{\mathbf{Q}}(\mathbf{q}) d\mathbf{q} \\ &= \int_{R(\mathbf{Q})} \left( \int_{R(\mathbf{X})} D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) \frac{f_{\mathbf{Q}|\mathbf{X}}(\mathbf{q}|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{f_{\mathbf{Q}}(\mathbf{q})} d\mathbf{x} \right) f_{\mathbf{Q}}(\mathbf{q}) d\mathbf{q} \\ &= \int_{R(\mathbf{Q})} \left( \int_{R(\mathbf{X})} D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) f_{\mathbf{Q}|\mathbf{X}}(\mathbf{q}|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \right) d\mathbf{q} \\ &= \int_{R(\mathbf{Q})} \left( \int_{R(\mathbf{X})} D(\hat{\mathbf{x}}_{opt}, \mathbf{x}) \frac{1}{\sqrt{(2\pi\mu)^N}} \exp\left(-\frac{\|\mathbf{q} - \mathbf{x}\|^2}{2\mu}\right) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \right) d\mathbf{q}. \end{aligned}$$

Using both claims, we further analyze the estimation performance limits for three example error metrics in Sections III-C.

### C. Examples

1) *Absolute error:* Because the MMSE is the mean of the conditional distribution, the outliers in the set of data may corrupt the estimation, and in this case the minimum mean absolute error (MMAE) is a good alternative. For absolute error,  $d_{AE}(\hat{x}_j, x_j) = |\hat{x}_j - x_j|$ , and we have the following corollary describing the performance limit of an MMAE estimator, where the proof is given in Appendix A.

*Corollary 1:* For a system modeled by (1), (2), the minimum mean absolute error (MMAE) estimator achieves

$$MMAE(f_{\mathbf{X}}, \mu) = N \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{\hat{x}_{j, MMAE}} (-x_j) f_{X_j|Q_j}(x_j|q_j) dx_j + \int_{\hat{x}_{j, MMAE}}^{+\infty} x_j f_{X_j|Q_j}(x_j|q_j) dx_j \right) f_{Q_j}(q_j) dq_j, \quad (10)$$

where  $x_j$  (respectively,  $q_j$ ) is the input (respectively, output) of the scalar Gaussian channel (5),  $\hat{x}_{j, MMAE}$  satisfies  $\int_{\hat{x}_{j, MMAE}}^{+\infty} f_{X_j|Q_j}(x_j|q_j) dx_j = \frac{1}{2}$ , and  $f_{X_j|Q_j}(x_j|q_j)$  is a function of  $f_{X_j}$  following (6).

2) *Support recovery error:* In some applications in compressed sensing, correctly estimating the locations where the data has non-zero values is almost as important as estimating the exact values of the data; it is a standard model selection error criterion [23]. The process of estimating the non-zero locations is called *support recovery*. Support

recovery error is defined as follows, and this metric function is discrete,

$$d_{\text{support}}(\hat{x}_j, x_j) = \text{xor}(x_j, \hat{x}_j),$$

where

$$\text{xor}(x_j, \hat{x}_j) = \begin{cases} 0, & \text{if } x_j = 0 \text{ and } \hat{x}_j = 0 \\ 0, & \text{if } x_j \neq 0 \text{ and } \hat{x}_j \neq 0 \\ 1, & \text{if } x_j = 0 \text{ and } \hat{x}_j \neq 0 \\ 1, & \text{if } x_j \neq 0 \text{ and } \hat{x}_j = 0 \end{cases}.$$

*Corollary 2:* For a system modeled by (1), (2), where  $f_{\mathbf{X}}$  is an i.i.d. sparse Gaussian prior such that  $\Pr(X_j \neq 0) = p$  and  $X_j \neq 0 \sim \mathcal{N}(0, \sigma^2)$ , the minimum mean support error (MMSuE) estimator achieves

$$\text{MMSuE}(f_{\mathbf{X}}, \mu) = N \cdot (1 - p) \cdot \text{erfc} \left( \sqrt{\frac{\tau}{2\mu}} \right) + N \cdot p \cdot \text{erf} \left( \sqrt{\frac{\tau}{2(\sigma^2 + \mu)}} \right), \quad (11)$$

where

$$\tau = 2 \cdot \frac{\sigma^2 + \mu}{\sigma^2/\mu} \cdot \ln \left( \frac{(1 - p)\sqrt{\sigma^2/\mu + 1}}{p} \right).$$

Corollary 2 is proved in Appendix B.

3) *Weighted-support error:* In Section III-C2, we put equal weights on the error patterns (i)  $\hat{x}_j \neq 0$  while  $x_j = 0$ ; and (ii)  $\hat{x}_j = 0$  while  $x_j \neq 0$ . In this section, we further put unequal weights on these two error patterns, and the *receiver operating characteristic* (ROC) curve [8] is obtained. We first define *false positive error*  $d_{\text{FP}}$  as

$$d_{\text{FP}}(\hat{x}_j, x_j) = d(\hat{x}_j = 1, x_j = 0) = 1,$$

and *false negative error*  $d_{\text{FN}}$  as

$$d_{\text{FN}}(\hat{x}_j, x_j) = d(\hat{x}_j = 0, x_j = 1) = 1.$$

Else the patterns coincide,

$$d(\hat{x}_j = 0, x_j = 0) = d(\hat{x}_j = 1, x_j = 1) = 0.$$

We then define *weighted-support error* as

$$d_{\text{w\_support}}(\hat{x}_j, x_j) = \beta \cdot d_{\text{FP}}(\hat{x}_j, x_j) + (1 - \beta) \cdot d_{\text{FN}}(\hat{x}_j, x_j), \quad (12)$$

where  $0 \leq \beta \leq 1$ , and we set  $d_{\text{w\_support}}(\hat{x}_j, x_j)$  as the error metric that we want to minimize. The false positive rate (or *false alarm rate*) is defined as  $\Pr(\hat{x}_j \neq 0 | x_j = 0)$ , and the false negative rate (or *misdetection rate*) is defined as  $\Pr(\hat{x}_j = 0 | x_j \neq 0)$ .

*Corollary 3:* For a system modeled by (1), (2), where  $f_{\mathbf{X}}$  is an i.i.d. sparse Gaussian prior such that  $\Pr(X_j \neq 0) = p$  and  $X_j \neq 0 \sim \mathcal{N}(0, \sigma^2)$ ,

1) the minimum mean weighted-support error (MMWSE) estimator achieves

$$\text{MMWSE}(f_{\mathbf{X}}, \mu) = N\beta(1 - p) \cdot \text{erfc} \left( \sqrt{\frac{\tau'}{2\mu}} \right) + N(1 - \beta)p \cdot \text{erf} \left( \sqrt{\frac{\tau'}{2(\sigma^2 + \mu)}} \right), \quad (13)$$



where

$$\tau' = 2 \cdot \frac{\sigma^2 + \mu}{\sigma^2/\mu} \cdot \ln \left( \frac{\beta(1-p)\sqrt{\sigma^2/\mu + 1}}{(1-\beta)p} \right).$$

2) The false positive rate is

$$\Pr(\hat{x}_j \neq 0 | x_j = 0) = \operatorname{erfc} \left( \sqrt{\frac{\tau'}{2\mu}} \right), \quad (14)$$

and the false negative rate is

$$\Pr(\hat{x}_j = 0 | x_j \neq 0) = \operatorname{erf} \left( \sqrt{\frac{\tau'}{2(\sigma^2 + \mu)}} \right). \quad (15)$$

The proof of Corollary 3 is provided in Appendix C.

It is shown in Corollary 3 that  $\Pr(\hat{x}_j \neq 0 | x_j = 0)$  and  $\Pr(\hat{x}_j = 0 | x_j \neq 0)$  vary when the value of  $\beta$  varies. Moreover, Comparing equation (13) to (11) in Corollary 2, the only difference is that  $\tau$  is replaced by  $\tau'$ , which is the *decision threshold* that determines whether an estimand is zero or nonzero. That said, putting different weights on false positive error and false negative error is analogous to tuning the decision threshold, and thus trading off between the false alarm rate and the misdetection rate [8]. An ROC curve is shown in Section IV.

#### IV. NUMERICAL RESULTS

Some numerical results are shown in this section to illustrate the performance of our estimation algorithm when minimizing a user-defined error metric. The Matlab implementation of our algorithm can be found at [http://people.engr.ncsu.edu/dzbaron/software/arb\\_metric/](http://people.engr.ncsu.edu/dzbaron/software/arb_metric/).

We test our estimation algorithm on two linear systems modeled by (1) and (2): (i) Gaussian input and Gaussian channel; (ii) Weibull input and Poisson channel. In both cases, the input's length  $N$  is 10,000, and its sparsity rate is 3%, meaning that the entries of the input vector are non-zero with probability 3%, and zero otherwise. The matrix  $\Phi$  we use is Bernoulli(0.5) distributed, and is normalized to have unit-norm rows. In the first case, the non-zero input entries are  $\mathcal{N}(0, 1)$  distributed, and the Gaussian noise is  $\mathcal{N}(0, 3 \cdot 10^{-4})$  distributed, i.e., the signal to noise ratio (SNR) is 20dB. In the second case, the non-zero input entries are Weibull distributed,

$$f(x_j; \lambda, k) = \begin{cases} \frac{k}{\lambda} \left(\frac{x_j}{\lambda}\right)^{k-1} e^{-(x_j/\lambda)^k} & x_j \geq 0 \\ 0 & x_j < 0 \end{cases},$$

where  $\lambda = 1$  and  $k = 0.5$ . The Poisson channel is

$$f_{Y|W}(y_i | w_i) = \frac{(\alpha w_i)^{y_i} e^{-(\alpha w_i)}}{y_i!}, \quad \text{for all } i \in \{1, 2, \dots, M\},$$

where the scaling factor of the input is  $\alpha = 100$ .

In order to illustrate that our estimation algorithm is suitable for reasonable error metrics, we considered absolute error and two other non-standard metrics:

$$\text{Error}_p = \sum_{j=1}^N |\hat{x}_j - x_j|^p,$$

where  $p = 0.5$  or  $1.5$ .

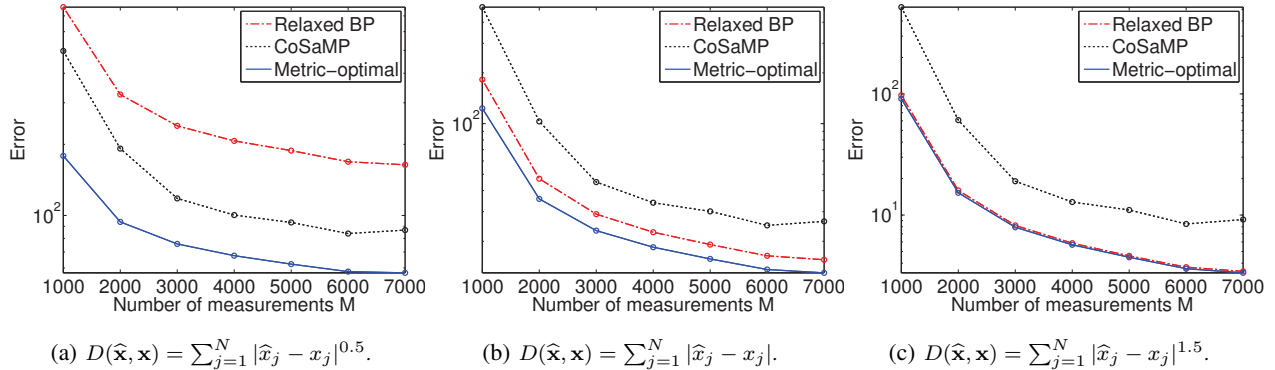


Fig. 3: Comparison of the metric-optimal estimation algorithm, relaxed BP, and CoSaMP. (Sparse Gaussian input and Gaussian channel; sparsity rate= 3%; input length  $N=10,000$ ; SNR=20dB.)

We compare our algorithm with the relaxed BP [6] and CoSaMP [19] algorithms. In Figure 3 and Figure 4, lines marked with “metric-optimal” present the errors of our estimation algorithm, and lines marked with “Relaxed BP” (respectively, “CoSaMP”) show the errors of the relaxed BP (respectively, CoSaMP) algorithm. Each point in the figure is an average of 100 experiments with the same parameters. Because the Poisson channel is not an additive noise channel and is not suitable for CoSaMP, the “MAE” and the “Error<sub>1.5</sub>” lines for “CoSaMP” in Figure 4 appear beyond the scope of vertical axis. It can be seen that our metric-optimal algorithm outperforms the other two methods.

To demonstrate the theoretical analysis of our algorithm in Sections III-C, we compare our MMAE estimation results with the theoretical limit (10) in Figure 5a, where the integrations are computed numerically. In Figure 5b, we compare our MMSuE estimator with the theoretical limit (11), where the value of  $\mu$  is acquired numerically from the relaxed BP method [35] with 20 iterations. In Figure 5c, our MMWSE estimator and its theoretical limit (13) are compared, where we fix the weight  $\beta = 0.3$ , and obtain the value of  $\mu$  as in Figure 5b. In all three figures, each point on the “metric-optimal” line is generated by averaging 40 experiments with the same parameters. It is shown from all figures that the two lines are on top of each other. Therefore our estimation algorithm reaches the corresponding theoretical limits and is optimal.

Figure 6 illustrates the ROC curve obtained by setting the weighted-support error (12) as the error metric. We vary the value of  $\beta$  in (12) from 0 to 1, and compute the false positive rate as well as the false negative rate from (14) and (15). The ROC curve is a  $\Pr(\hat{x}_j \neq 0 | x_j = 0)$  (14) versus  $\Pr(\hat{x}_j \neq 0 | x_j \neq 0)$  plot, where  $\Pr(\hat{x}_j \neq 0 | x_j \neq 0)$  is called the *true positive rate*, and  $\Pr(\hat{x}_j \neq 0 | x_j \neq 0) = 1 - \Pr(\hat{x}_j = 0 | x_j \neq 0)$  (15). In order to obtain different curves, we tune the number of measurements  $M$ , while keeping the sparsity rate, input length, and the SNR fixed. It can be seen that for the same level of false positive rate, a greater number of measurements achieves a higher true positive rate.

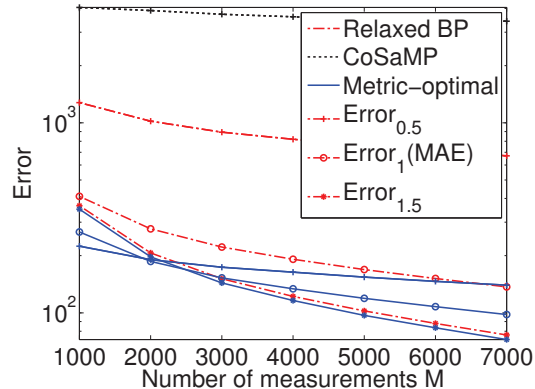


Fig. 4: Comparison of the metric-optimal estimation algorithm, relaxed BP, and CoSaMP. The “MAE” and the “Error<sub>1.5</sub>” lines for “CoSaMP” appear beyond the scope of vertical axis. (Sparse Weibull input and Poisson channel; sparsity rate= 3%; input length  $N=10,000$ ; input scaling factor  $\alpha = 100$ .)

## V. CONCLUSION

In this correspondence, we introduced a pointwise estimation algorithm that deals with arbitrary additive error metrics in noisy compressed sensing. We verified that the algorithm is optimal in the large system limit, and provided a general method to compute the minimum expected error achievable by any estimation algorithm for a user-defined additive error metric. We started with the scalar Gaussian channel model of the relaxed BP algorithm and extended it to a method that is applicable to any user-defined additive error metric. We discussed three error metric examples, absolute error, support error, and weighted-support error, and gave the theoretical performance limits for them. We further obtained the ROC curve for the modeled system by minimizing the weighted-support error. We also illustrated numerically that our algorithm reaches the three example theoretical limits, and outperforms the relaxed BP and the CoSaMP methods.

## APPENDIX A

### PROOF OF COROLLARY 1

When

$$d_{\text{AE}}(\hat{x}_j, x_j) = |\hat{x}_j - x_j|,$$

equations (7) or (8) solve for the MMAE estimand,  $\hat{x}_{\text{MMAE}}$ . In order to find the minimum, we take the derivative of the expected function over  $\hat{x}_j$ ,

$$\left. \frac{d E(|\hat{x}_j - x_j| | q_j)}{d \hat{x}_j} \right|_{\hat{x}_j = \hat{x}_{j, \text{MMAE}}} = 0, \quad (16)$$

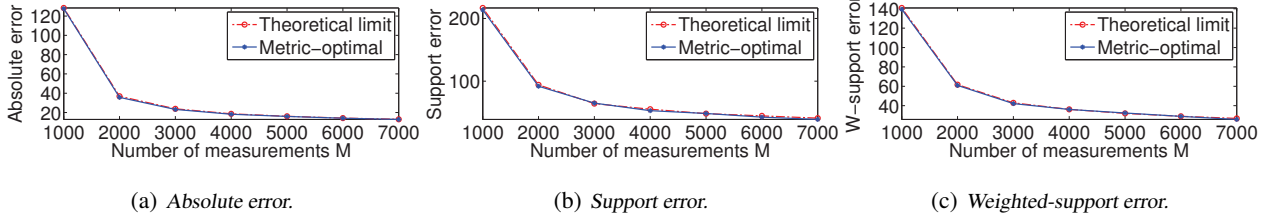


Fig. 5: Comparisons of the metric-optimal estimators and the corresponding theoretical limits (10), (11), and (13). The corresponding two lines are on top of each other. (Sparse Gaussian input and Gaussian channel; sparsity rate= 3%; input length  $N=10,000$ ; SNR=20dB.)

for each  $j = \{1, 2, \dots, N\}$ . But

$$\begin{aligned}
E(|\hat{x}_j - x_j| | q_j) &= \int_0^\infty \Pr(|\hat{x}_j - x_j| > t | q_j) dt \\
&= \int_0^\infty \Pr(\hat{x}_j - x_j > t | q_j) dt + \int_0^\infty \Pr(\hat{x}_j - x_j < -t | q_j) dt \\
&= \int_{-\infty}^{\hat{x}_j} \Pr(x_j < t_1 | q_j) dt_1 + \int_{\hat{x}_j}^\infty \Pr(x_j > t_2 | q_j) dt_2,
\end{aligned} \tag{17}$$

where changes of variables  $t_1 = \hat{x}_j - t$  and  $t_2 = \hat{x}_j + t$  are applied in (17). Using (16) and (17), we need

$$\Pr(x_j < \hat{x}_{j,\text{MMAE}} | q_j) - \Pr(x_j > \hat{x}_{j,\text{MMAE}} | q_j) = 0,$$

and thus  $\hat{x}_{j,\text{MMAE}}$  is given as the median of the conditional statistics  $f_{X_j|Q_j}(x_j|q_j)$ ,

$$\int_{-\infty}^{\hat{x}_{j,\text{MMAE}}} f_{X_j|Q_j}(x_j|q_j) dx_j = \int_{\hat{x}_{j,\text{MMAE}}}^{+\infty} f_{X_j|Q_j}(x_j|q_j) dx_j = \frac{1}{2}.$$

Then, the conditional mean absolute error is,

$$\begin{aligned}
&E[|\hat{x}_{j,\text{MMAE}} - x_j| | q_j] \\
&= \int_{-\infty}^{+\infty} |\hat{x}_{j,\text{MMAE}} - x_j| f_{X_j|Q_j}(x_j|q_j) dx_j \\
&= \int_{-\infty}^{\hat{x}_{j,\text{MMAE}}} (\hat{x}_{j,\text{MMAE}} - x_j) f_{X_j|Q_j}(x_j|q_j) dx_j + \int_{\hat{x}_{j,\text{MMAE}}}^{+\infty} (x_j - \hat{x}_{j,\text{MMAE}}) f_{X_j|Q_j}(x_j|q_j) dx_j \\
&= \int_{-\infty}^{\hat{x}_{j,\text{MMAE}}} (-x_j) f_{X_j|Q_j}(x_j|q_j) dx_j + \int_{\hat{x}_{j,\text{MMAE}}}^{+\infty} x_j f_{X_j|Q_j}(x_j|q_j) dx_j.
\end{aligned}$$

Therefore, the MMAE for location  $j$ ,  $\text{MMAE}_j(f_{X_j}, \mu)$ , is

$$\begin{aligned}
\text{MMAE}_j(f_{X_j}, \mu) &= E[|\hat{x}_{j,\text{MMAE}} - x_j|] \\
&= \int_{-\infty}^{+\infty} E[|\hat{x}_{j,\text{MMAE}} - x_j| | q_j] f_{Q_j}(q_j) dq_j \\
&= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{\hat{x}_{j,\text{MMAE}}} (-x_j) f_{X_j|Q_j}(x_j|q_j) dx_j + \int_{\hat{x}_{j,\text{MMAE}}}^{+\infty} x_j f_{X_j|Q_j}(x_j|q_j) dx_j \right) f_{Q_j}(q_j) dq_j.
\end{aligned}$$

We note in passing that the integrations can be evaluated numerically in an implementation.

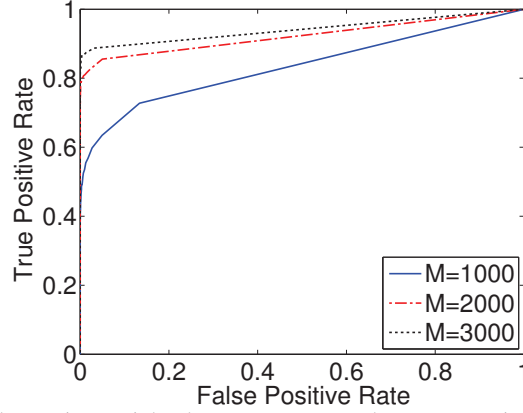


Fig. 6: The ROC curve obtained by setting weighted-support error as the error metric. (Sparse Gaussian input and Gaussian channel; sparsity rate= 3%; input length  $N=10,000$ ; SNR=20dB.)

Because the input  $\mathbf{x}$  is i.i.d., and the decoupled scalar channels have the same parameter  $\mu$ , the values of  $\text{MMAE}_j$  for all  $j \in \{1, 2, \dots, N\}$  are the same, and the overall MMAE is

$$\text{MMAE}(f_{\mathbf{x}}, \mu) = N \cdot \text{MMAE}_j(f_{X_j}, \mu).$$

## APPENDIX B

### PROOF OF COROLLARY 2

Similar to the idea of giving a limit on support recovery error rate [24], we derive the MMSuE limit for the case where the input is real-valued and the matrix  $\Phi$  is rectangular ( $M < N$ ). In the scalar Gaussian channel (5), we factor the sparse Gaussian input  $X_j$  into  $X_j = U_j \cdot B_j$ , where  $U_j \sim \mathcal{N}(0, \sigma^2)$  and  $B_j \sim \text{Bernoulli}(p)$ , i.e.,  $\Pr(B_j = 1) = p = 1 - \Pr(B_j = 0)$ . The support recovery problem is the task of finding the maximum a-posteriori (MAP) estimation of  $B_j$ .

For our estimation algorithm, the conditional expectation of support recovery error is,

$$E[d(\hat{x}_j, x_j)|q_j] = \begin{cases} \Pr(B_j = 1|q_j) & \text{if } \hat{b}_j = 0 \text{ and } b_j = 1 \\ \Pr(B_j = 0|q_j) & \text{if } \hat{b}_j = 1 \text{ and } b_j = 0 \\ 0 & \text{if } \hat{b}_j = 0 \text{ and } b_j = 0 \\ 0 & \text{if } \hat{b}_j = 1 \text{ and } b_j = 1 \end{cases}.$$

The estimand  $\hat{b}_{j,\text{opt}}$  minimizes  $E[d(\hat{x}_j, x_j)|q_j]$ , which implies

$$\hat{b}_{j,\text{opt}} = \begin{cases} 0 & \text{if } \Pr(B_j = 1|q_j) \leq \Pr(B_j = 0|q_j) \\ 1 & \text{if } \Pr(B_j = 1|q_j) > \Pr(B_j = 0|q_j) \end{cases}. \quad (18)$$

It is easy to see that  $f_{Q_j|B_j}(q_j|0) \sim \mathcal{N}(0, \mu)$  and  $f_{Q_j|B_j}(q_j|1) \sim \mathcal{N}(0, \sigma^2 + \mu)$ . Then,

$$\begin{aligned} \Pr(B_j = 1|q_j) &= \frac{f_{Q_j|B_j}(q_j|1)\Pr(B_j = 1)}{f_{Q_j}(q_j)} \\ &= \frac{f_{Q_j|B_j}(q_j|1)\Pr(B_j = 1)}{\sum_{b_j=0,1} f_{Q_j|B_j}(q_j|b_j)\Pr(B_j = b_j)} \\ &= \frac{1}{1 + \frac{1-p}{p} \sqrt{\sigma^2/\mu + 1} \exp\left(-\frac{q_j^2}{2} \cdot \frac{\sigma^2/\mu}{\sigma^2+\mu}\right)}, \end{aligned} \quad (19)$$

and similarly

$$\begin{aligned} \Pr(B_j = 0|q_j) &= \frac{f_{Q_j|B_j}(q_j|0)\Pr(B_j = 0)}{f_{Q_j}(q_j)} \\ &= \frac{\frac{1-p}{p} \sqrt{\sigma^2/\mu + 1} \exp\left(-\frac{q_j^2}{2} \cdot \frac{\sigma^2/\mu}{\sigma^2+\mu}\right)}{1 + \frac{1-p}{p} \sqrt{\sigma^2/\mu + 1} \exp\left(-\frac{q_j^2}{2} \cdot \frac{\sigma^2/\mu}{\sigma^2+\mu}\right)}. \end{aligned} \quad (20)$$

Therefore,  $\Pr(B_j = 1|q_j) > \Pr(B_j = 0|q_j)$  implies

$$q_j^2 > \tau = 2 \cdot \frac{\sigma^2 + \mu}{\sigma^2/\mu} \ln\left(\frac{(1-p)\sqrt{\sigma^2/\mu + 1}}{p}\right),$$

and vice versa. We can rewrite (18) as,

$$\hat{b}_{j,\text{opt}} = \begin{cases} 0 & \text{if } q_j^2 \leq \tau \\ 1 & \text{if } q_j^2 > \tau \end{cases}.$$

By averaging over the range of  $Q_j$ , we get the overall MMSuE,

$$\begin{aligned} \text{MMSuE}(f_{\mathbf{X}}, \mu) &= N \cdot E[d_{j,\text{support}}(\hat{x}_j, x_j)] \\ &= N \int E(d_{j,\text{support}}(\hat{x}_j, x_j)|q_j) f_{Q_j}(q_j) dq_j \\ &= N \int_{q_j^2 > \tau} \Pr(B_j = 0|q_j) f_{Q_j}(q_j) dq_j + N \int_{q_j^2 \leq \tau} \Pr(B_j = 1|q_j) f_{Q_j}(q_j) dq_j \\ &= N \cdot \Pr(B_j = 0, q_j^2 > \tau) + N \cdot \Pr(B_j = 1, q_j^2 \leq \tau) \\ &= N \cdot \Pr(q_j^2 > \tau|B_j = 0)\Pr(B_j = 0) + N \cdot \Pr(q_j^2 \leq \tau|B_j = 1)\Pr(B_j = 1) \\ &= N(1-p) \cdot \text{erfc}\left(\sqrt{\frac{\tau}{2\mu}}\right) + Np \cdot \text{erf}\left(\sqrt{\frac{\tau}{2(\sigma^2 + \mu)}}\right). \end{aligned}$$

#### APPENDIX C

##### PROOF OF COROLLARY 3

We use the same variables  $U_j$  and  $B_j$  as defined in Appendix B. For  $d_{w,\text{support}}$  (12), its conditional expectation is

$$E[d_{w,\text{support}}(\hat{x}_j, x_j)|q_j] = \begin{cases} (1-\beta) \cdot \Pr(B_j = 1|q_j) & \text{if } \hat{b}_j = 0 \text{ and } b_j = 1 \\ \beta \cdot \Pr(B_j = 0|q_j) & \text{if } \hat{b}_j = 1 \text{ and } b_j = 0 \\ 0 & \text{if } \hat{b}_j = 0 \text{ and } b_j = 0 \\ 0 & \text{if } \hat{b}_j = 1 \text{ and } b_j = 1 \end{cases}.$$

The estimand  $\widehat{b}_{j,\text{opt}}$  minimizes  $E[d_{\text{w\_support}}(\widehat{x}_j, x_j)|q_j]$ , which implies

$$\widehat{b}_{j,\text{opt}} = \begin{cases} 0 & \text{if } (1 - \beta) \cdot \Pr(B_j = 1|q_j) \leq \beta \cdot \Pr(B_j = 0|q_j) \\ 1 & \text{if } (1 - \beta) \cdot \Pr(B_j = 1|q_j) > \beta \cdot \Pr(B_j = 0|q_j) \end{cases}. \quad (21)$$

Plugging (19) and (20) into (21), we get that

$$\widehat{b}_{j,\text{opt}} = \begin{cases} 0 & \text{if } q_j^2 \leq \tau' \\ 1 & \text{if } q_j^2 > \tau' \end{cases},$$

where

$$q_j^2 > \tau' = 2 \cdot \frac{\sigma^2 + \mu}{\sigma^2/\mu} \ln \left( \frac{\beta(1-p)\sqrt{\sigma^2/\mu + 1}}{(1-\beta)p} \right).$$

Therefore, the minimum mean weighted-support error function  $\text{MMWSE}(f_{\mathbf{X}}, \mu)$  is,

$$\begin{aligned} \text{MMWSE}(f_{\mathbf{X}}, \mu) &= N \cdot E[d_{j,\text{w\_support}}(\widehat{x}_j, x_j)] \\ &= N \int E(d_{j,\text{w\_support}}(\widehat{x}_j, x_j)|q_j) f_{Q_j}(q_j) dq_j \\ &= N \int_{q_j^2 > \tau'} \beta \Pr(B_j = 0|q_j) f_{Q_j}(q_j) dq_j + N \int_{q_j^2 \leq \tau'} (1 - \beta) \Pr(B_j = 1|q_j) f_{Q_j}(q_j) dq_j \\ &= N\beta(1-p) \cdot \text{erfc} \left( \sqrt{\frac{\tau'}{2\mu}} \right) + N(1-\beta)p \cdot \text{erf} \left( \sqrt{\frac{\tau'}{2(\sigma^2 + \mu)}} \right). \end{aligned}$$

The false positive rate is

$$\begin{aligned} \Pr(\widehat{x}_j \neq 0|x_j = 0) &= \Pr(q_j^2 > \tau'|x_j = 0) \\ &= \Pr(q_j^2 > \tau'|B_j = 0) \\ &= \text{erfc} \left( \sqrt{\frac{\tau'}{2\mu}} \right), \end{aligned}$$

and the false negative rate is

$$\begin{aligned} \Pr(\widehat{x}_j = 0|x_j \neq 0) &= \Pr(q_j^2 \leq \tau'|x_j \neq 0) \\ &= \Pr(q_j^2 \leq \tau'|B_j = 1) \\ &= \text{erf} \left( \sqrt{\frac{\tau'}{2(\sigma^2 + \mu)}} \right). \end{aligned}$$

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