Universal MAP Estimation in Compressed Sensing

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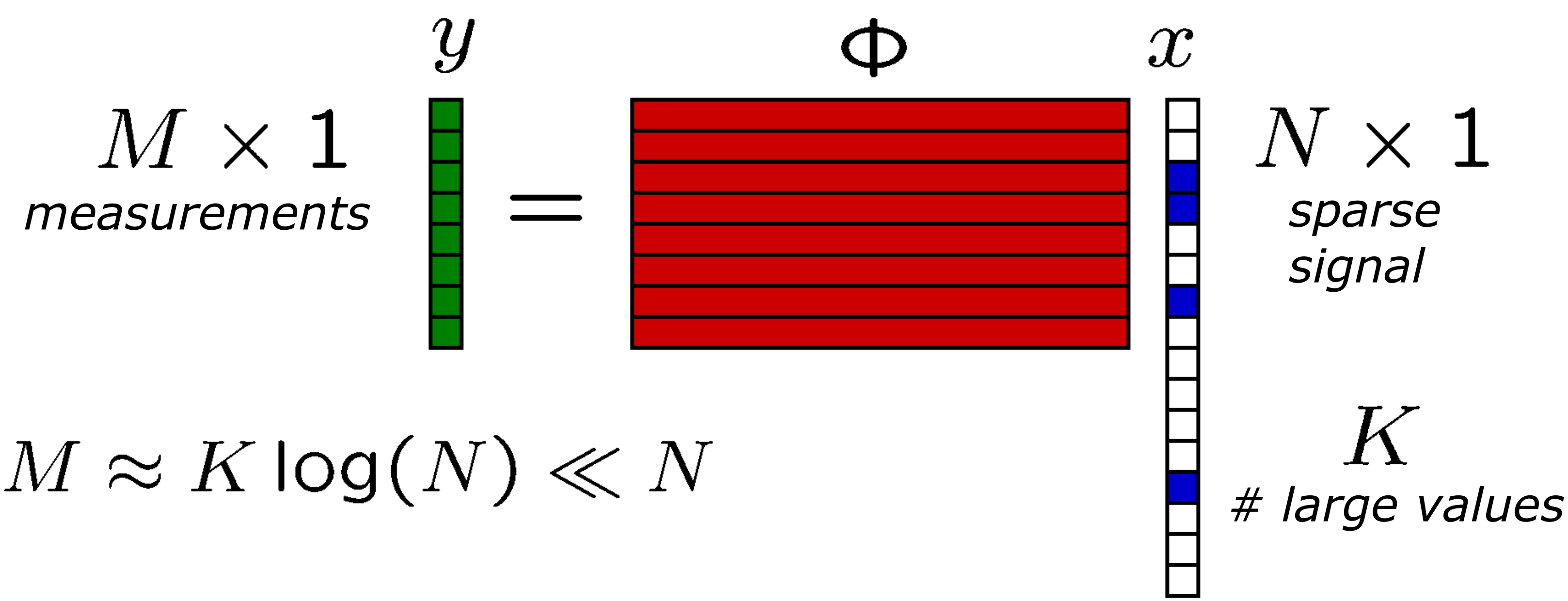
Compressed Sensing 101

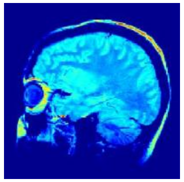
- Length-N input x, matrix Φ, K large coefficients
- $M \approx K \log(N) \ll N$ measurements suffice for robust signal reconstruction

$$y = \Phi x + z$$



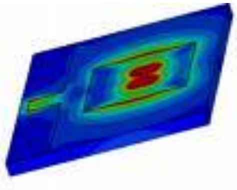
Estimation in Linear Mixing

- Medical imaging (tomography)



- Multiuser detection

- Financial prediction
- Electromagnetic scattering



- Seismic imaging (oil industry)

- Compressed sensing

- *Many more…*

Typically sparse x

or compressible x

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or compressible x

How about a <u>SIMPLE</u> x?

[Donoho et al. 2006]

Reconstructing *Simple* Source [N=10,000]

- x generated by four state Markov
- 1 1 -1 -1 1 1 -1 -1 1 1 -1 -1…
- 3% glitches in state transitions

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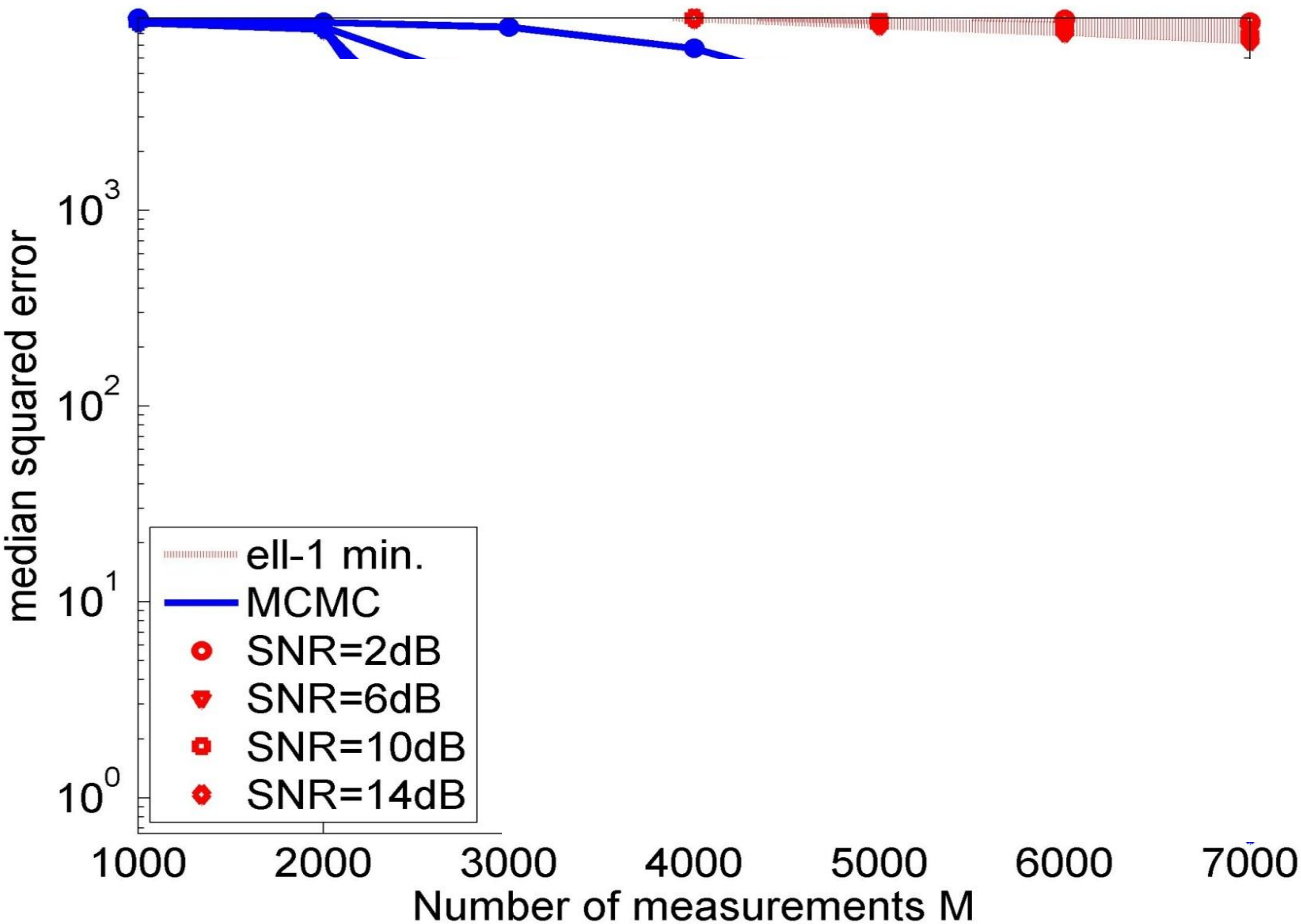

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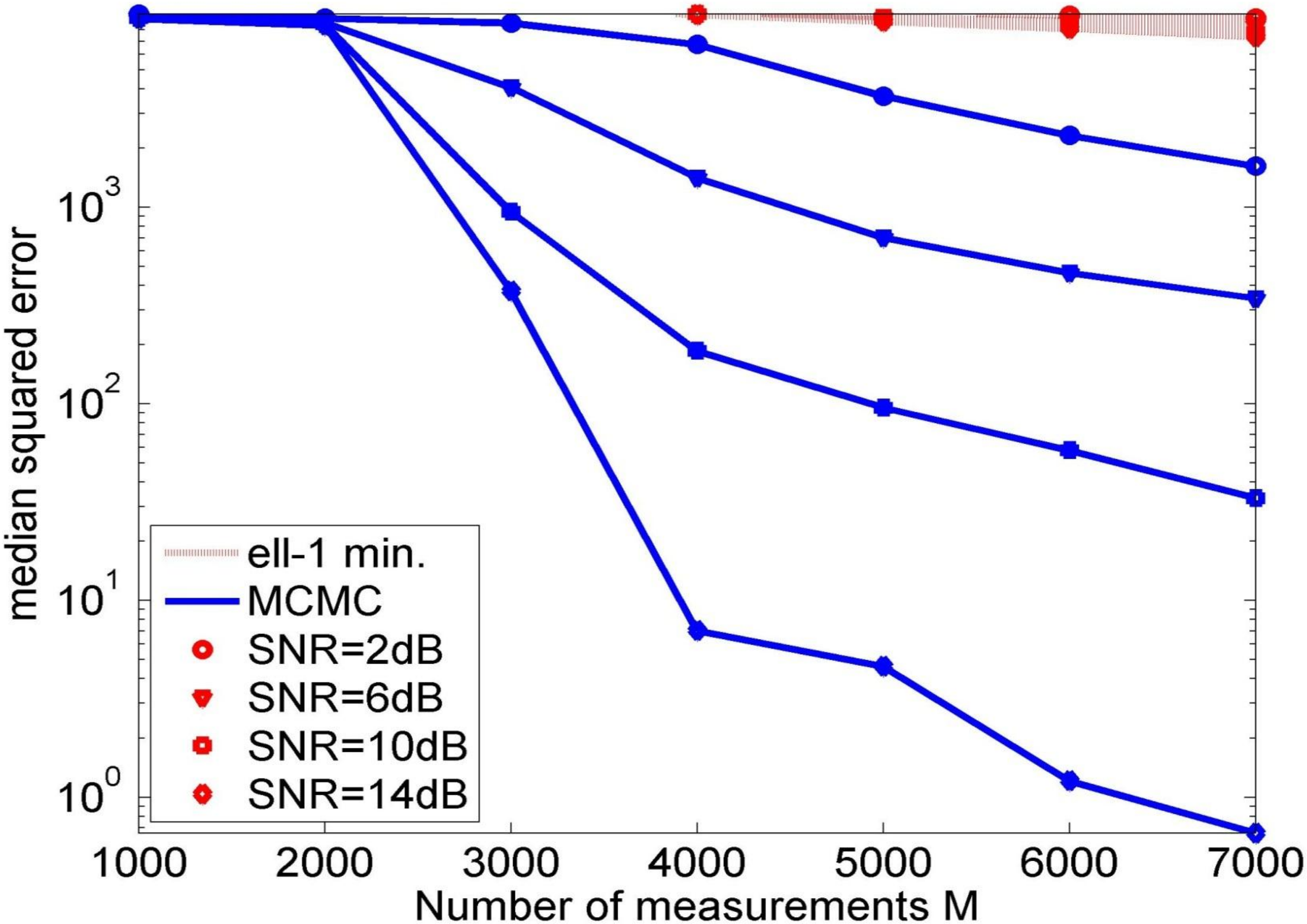


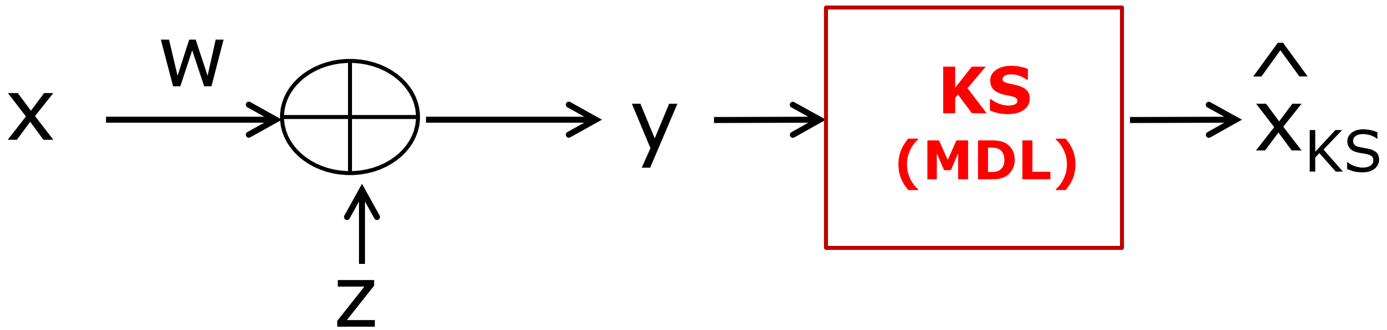
How Did We Do That???

Kolmogorov Sampler [Donoho 2002]

- AWGN *scalar channel* $y = x + z$
- Estimate x by minimizing for
  - *complexity* $K(x)$ [Kolmogorov, Rissanen,...]
  - regularized via log loss: $-\log(f_Z(Z=y-x))$

- Kolmogorov sampler = minimum description length

$$\hat{x}_{KS} = \operatorname{argmin}_{\hat{x}} \{K(\hat{x}) - \log(f_Z(Z=y-\hat{x}))\}$$



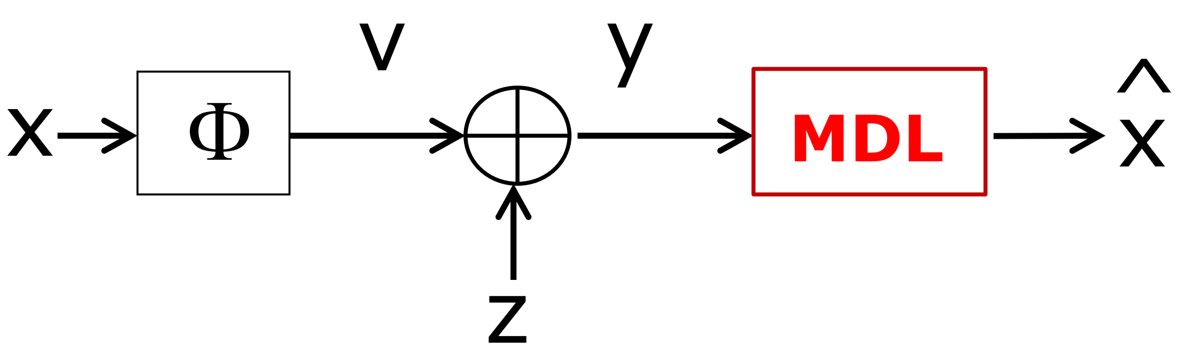
MDL in CS Estimation

- Estimator *loss/regret* $-\log(f_{Y|V}(Y=y|V=\Phi x))$

- MDL = maximum *a posteriori* w/ complexity prior

$$\hat{x}_{MDL} = \text{argmin}_{\hat{x}} \{K(\hat{x}) - \log(f_{Y|V}(Y=y|V=\Phi\hat{x}))\}$$

- Optimization over real-valued $\hat{x}$ ☹



Finite Optimization

- <u>**MAIN IDEAS**</u>

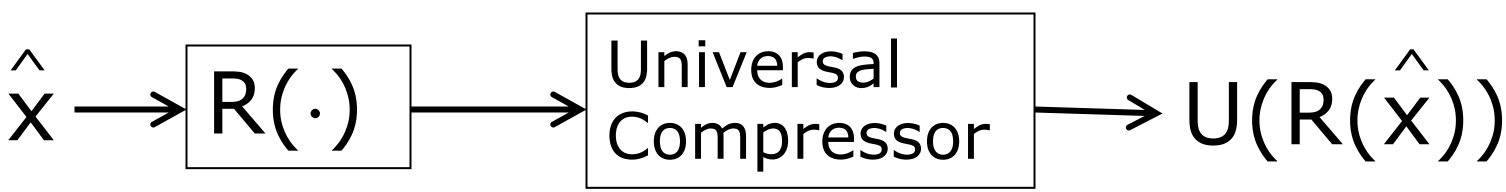
(1) *Quantize* with reproduction levels R

(2) *Encode* $R(\hat{x})$ with universal coding length $U(R(\hat{x}))$

  - details of U later

- <u>**Theorem**</u>: regret for discretized grid ε-close to regret over continuous space

- Replaced real-valued by finite optimization ☺



Is MDL Estimation Good?

- Scalar channel $\Phi=I$, $y=x+z$: $E[(x_{KS}-x)^2]$ is *double* the Bayesian minimum mean square error (MMSE)

- **<u>Conjecture</u>**: $E[(\hat{X}_{MDL}-x)^2]=2MMSE$ in $y=\Phi x+z$ channel

- **<u>Conjecture</u>**: ε-weaker performance using quantized grid and universal code $U(R(\hat{x}))$

- Double the MMSE is ***bad*** for low SNR

- Alternative - *mixture* over all possible $\hat{x}$ [Baron 2011]

Algorithmic Approach

Inspired by universal lossy data compression [Weissman et al.]

Coding Stationary Ergodic Sources

- Assume x generated by stationary ergodic source X
  - process $\hat{x}$ over quantized space
- Our ***practical encoder*** $U(\cdot)$ computes:
  - *empirical symbol counts* $n_q(\alpha,\beta,\hat{x})$
  - # times $\beta\in R$ appeared after *context* $\alpha\in R^q$ in $\hat{x}$
  - empirical conditional probabilities $p_q(\beta|\alpha,\hat{x})$
- Empirical conditional *entropy*

$$U(\hat{x}) = H_q(\hat{x}) = -\sum_{\alpha,\beta} n_q(\alpha,\beta,\hat{x}) \log(p_q(\beta|\alpha,\hat{x}))$$

- $H_q(\cdot)$ quantifies likelihood for unknown stationary ergodic X as $N\rightarrow\infty$

Markov Chain Monte Carlo (MCMC)

- Initialize $\hat{x}$ (over quantized space)
- Process one symbol $\hat{x}_i$ at a time

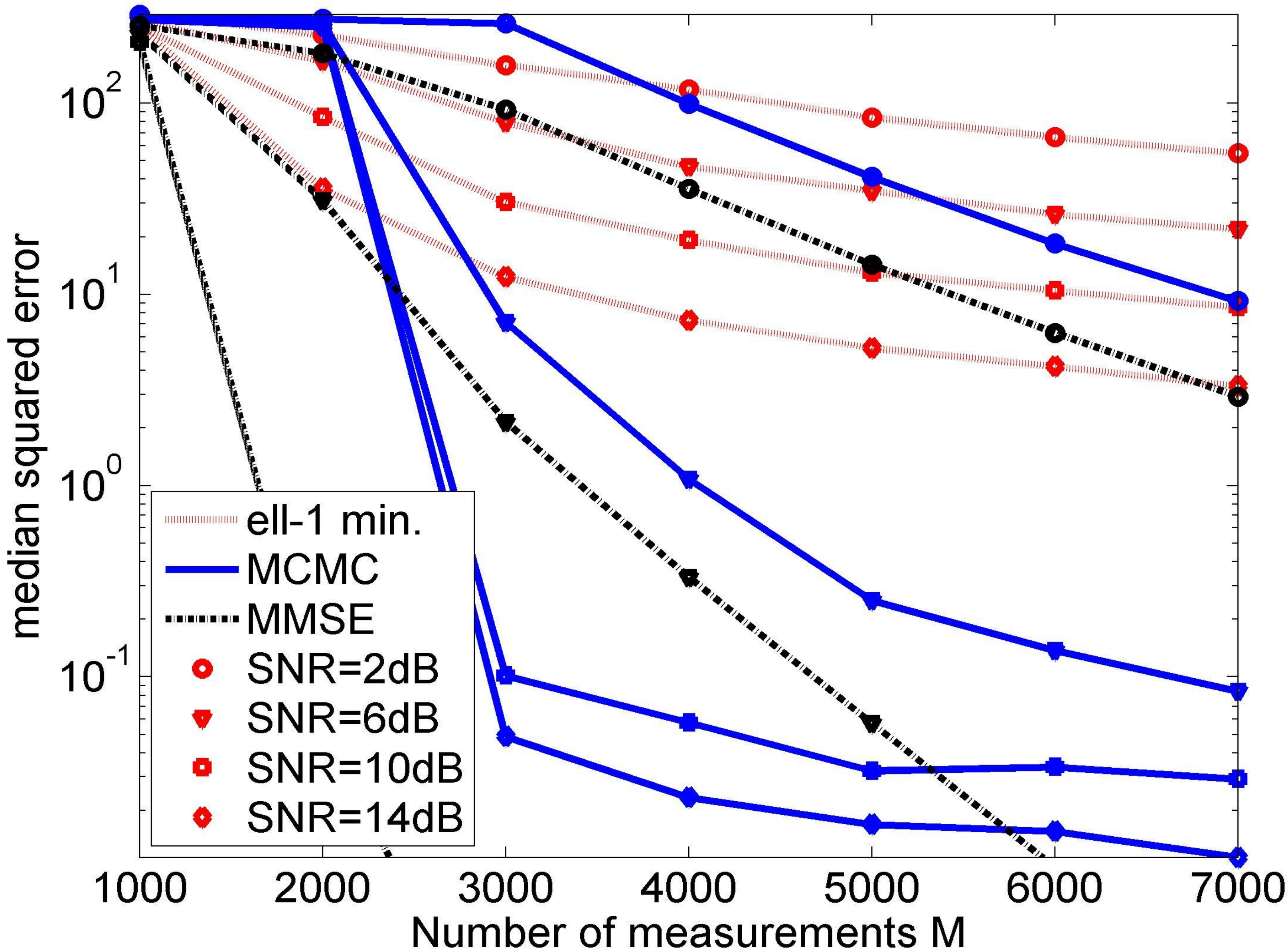
- Generate $\hat{x}_i$ randomly from Gibbs distribution

- Probability based on

$$Pr(\hat{x}) \propto \exp\{-\mathbf{s} \cdot (U(\hat{x}) - \log(f_{Y|W}(W=y|W=J(\hat{x})))$$

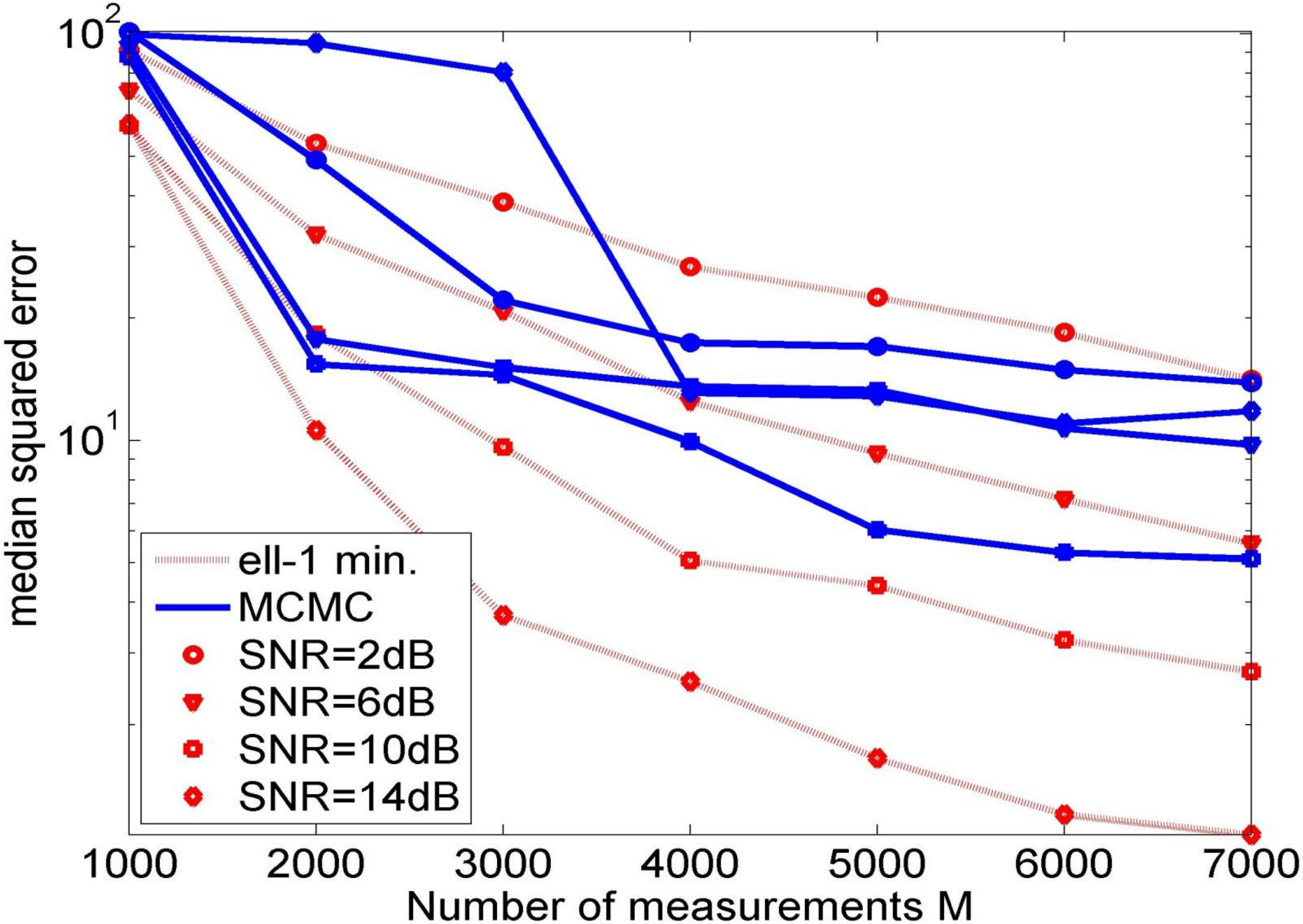
- Analogous to heat bath concepts in statistical physics
  - inverse temperature s
  - gradual cool-down

Performance [x Bernoulli, N=10,000]



Performance [N=10,000]

- Support set nonzero 3% of time
- Nonzero runs of length ∼ 10 times
- Uniform U[0,1] when nonzero



Challenges

- Strong dependence on initialization point
- Robust + adaptive reproduction levels R
- More sophisticated algorithms (mixtures?)
- Application-specific families of priors (e.g., images)
- Rigorous theoretical justification for MSE=2MMSE
- **Download our software!!**

people.engr.ncsu.edu/dzbaron/software/UCS_BaronDuarte

THE END