Signal Estimation with Low Infinity-Norm Error by Minimizing the Mean p-Norm Error

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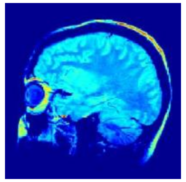
PRINCETON

March 19-21, 2014

Motivation

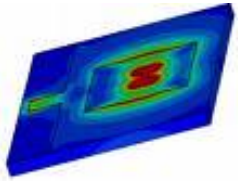
Signal Estimation

- Medical imaging (tomography)



- Source and channel coding

- Financial prediction



- Electromagnetic scattering

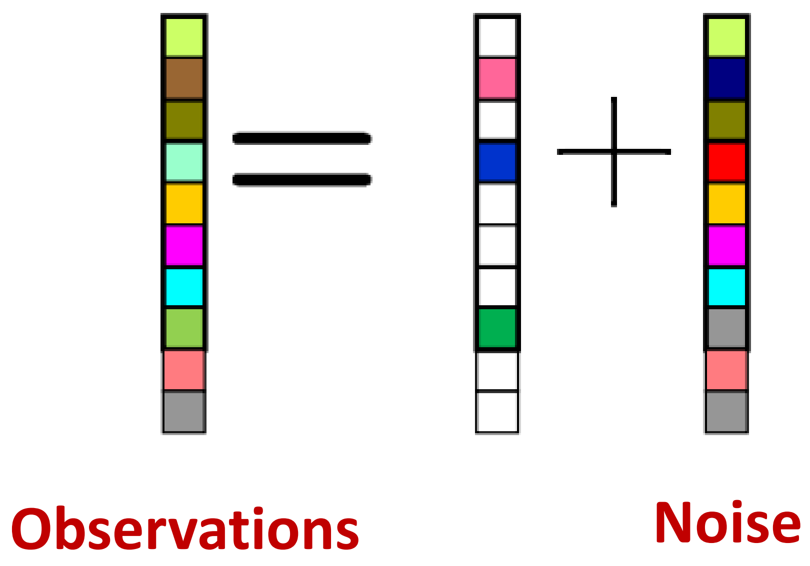
- Seismic imaging (oil industry)

- Speech recognition

- *Many more…*

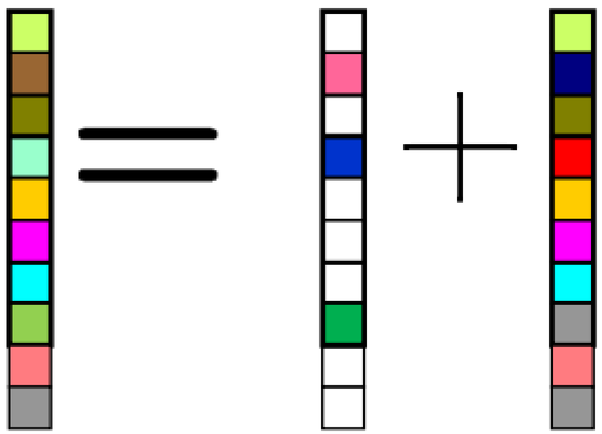
Noise introduced from sampling, transmission, compression and decompression, …

$$y = x + z$$



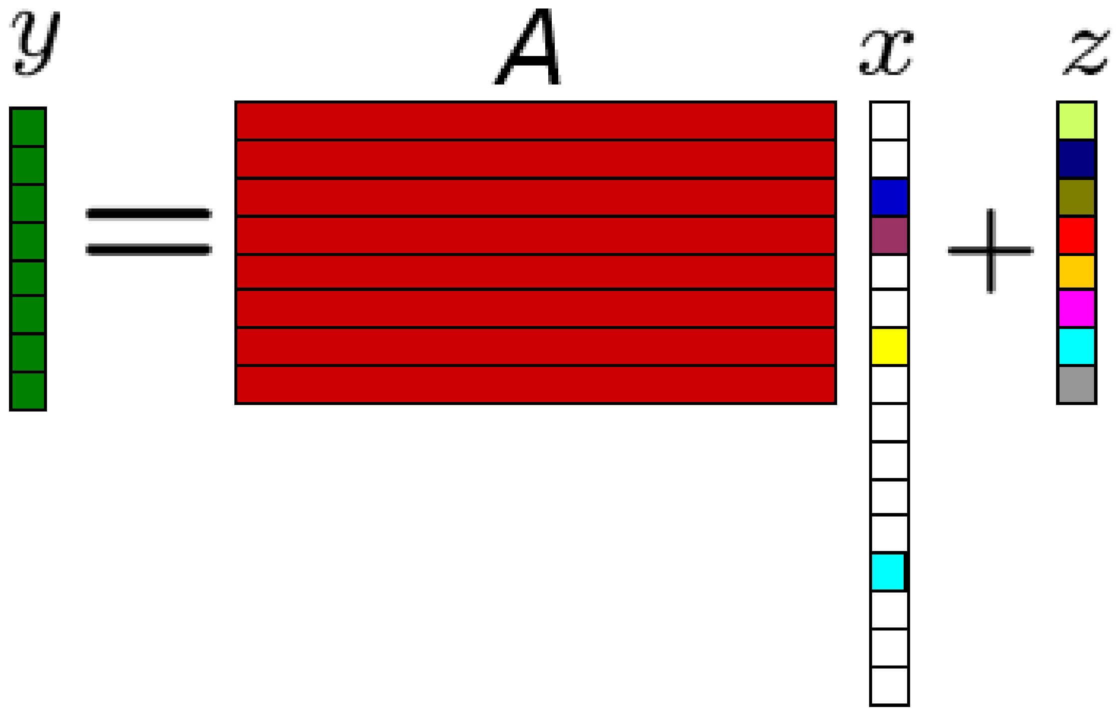
Noise introduced from sampling, transmission, compression and decompression, …

$$y = x + z$$ Scalar channel



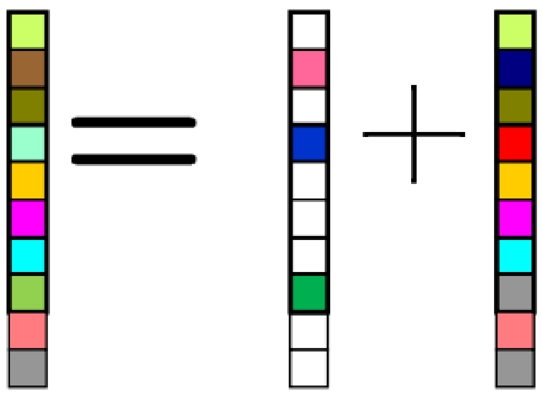
Observations **Noise**

$$y = Ax + z$$ Matrix channel



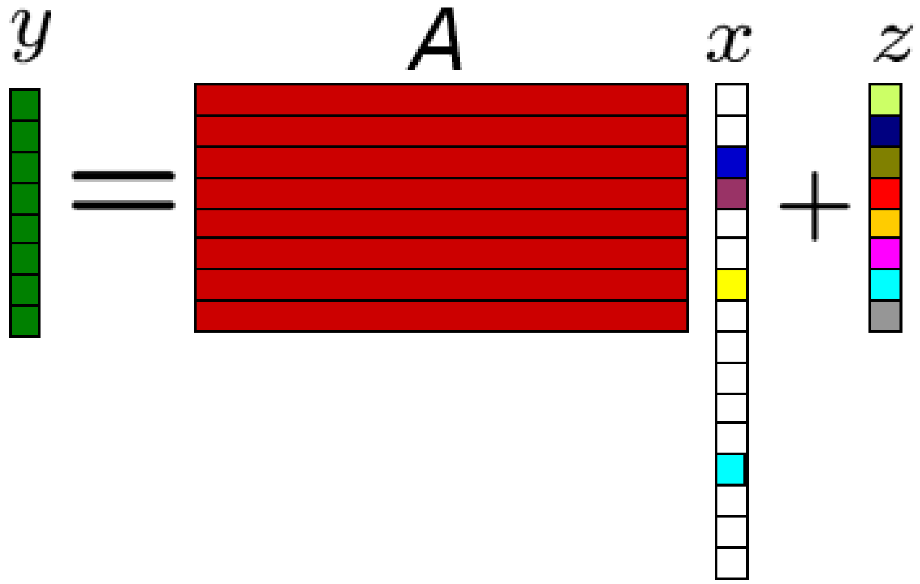
Scalar channels

$$y = x + z$$



Matrix channels

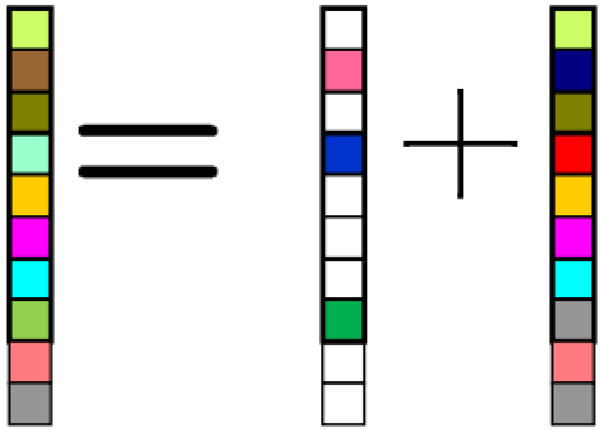
$$y = Ax + z$$



Scalar channels

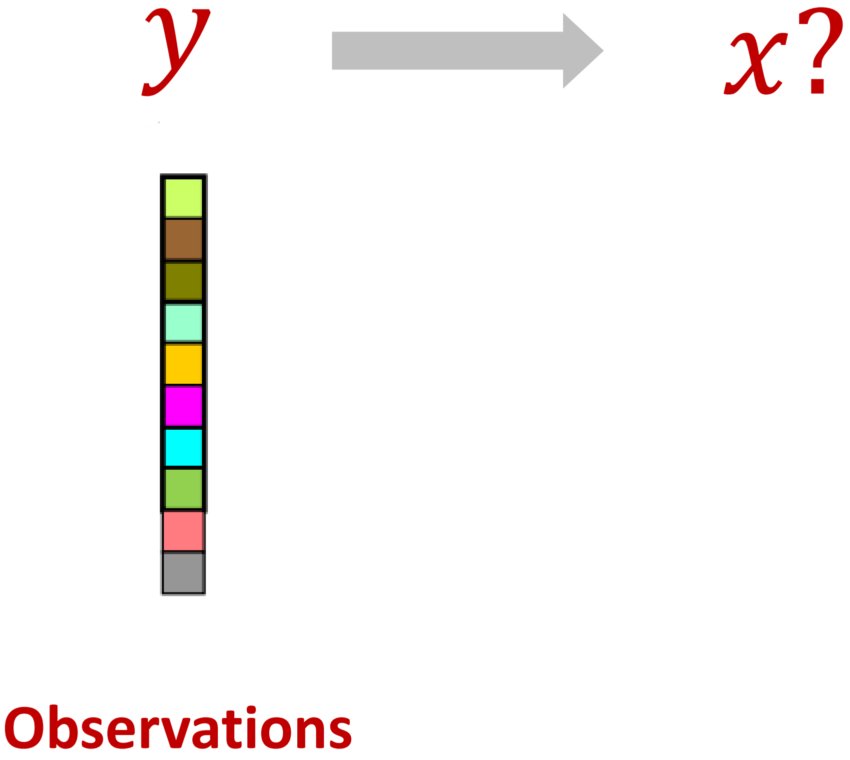
Scalar channels

$$y = x + z$$

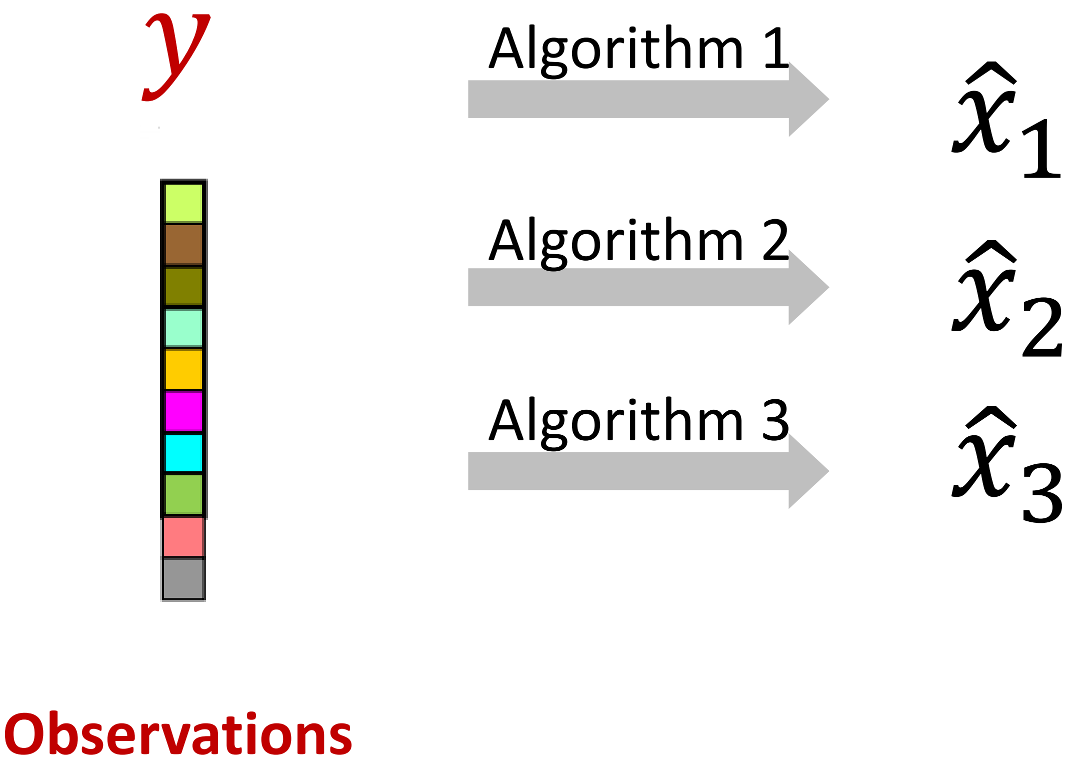


Observations **Noise**

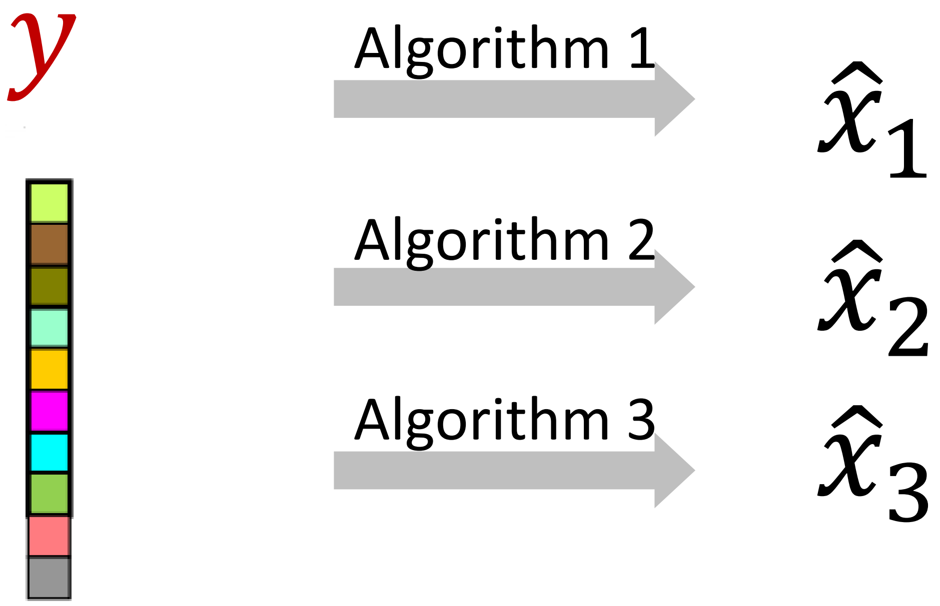
Scalar channels



Scalar channels



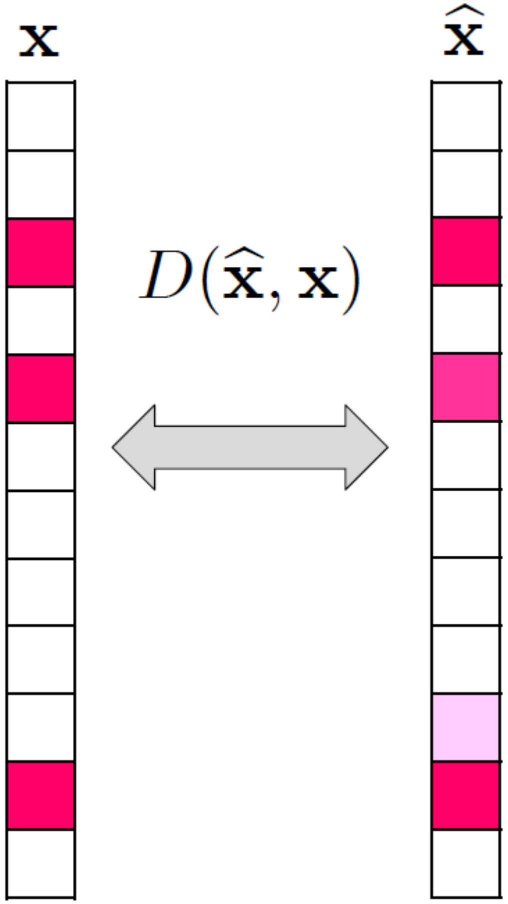
Scalar channels



Observations

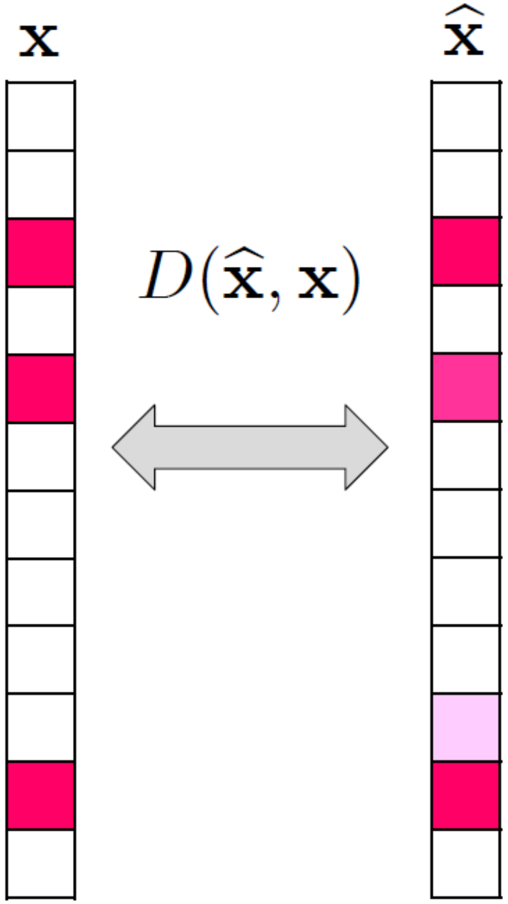
Which $\hat{x}$ is the **best** estimate?

Error metric



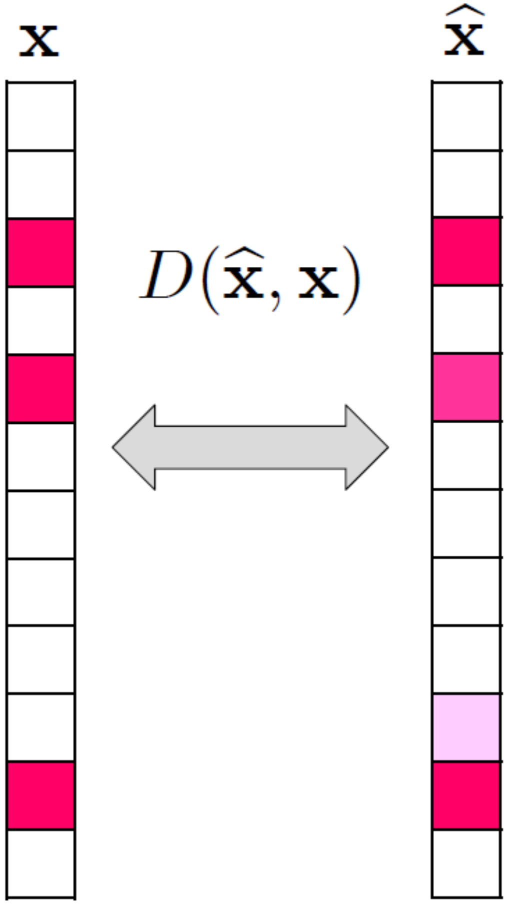
Examples:

- $||\hat{x} - x||_1$ (absolute error)
- $||\hat{x} - x||_2$ (square error)
- Hamming distance
- and more…



$$D(\hat{x}, x) = ||\hat{x} - x||_\infty$$
$$= \max_i |\hat{x}_i - x_i|$$

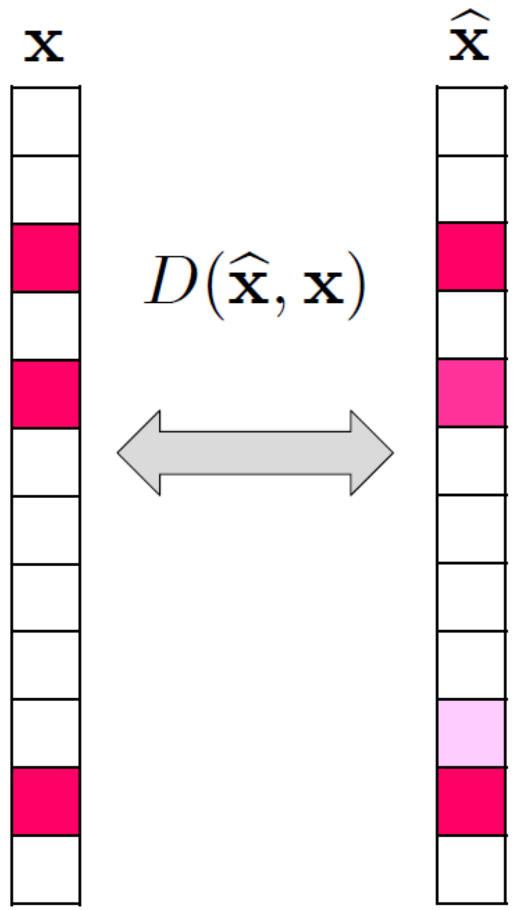
- Group testing
- Trajectory plan in control system
- OFDM



$$D(\hat{x}, x) = ||\hat{x} - x||_\infty$$

Expected error

$$\hat{x}_\infty = \underset{\hat{x}}{\operatorname{argmin}} E[\,||\hat{x} - x||_\infty | y]$$



$$D(\hat{x}, x) = ||\hat{x} - x||_\infty$$

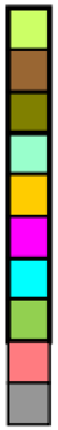
Expected error

$$\hat{x}_\infty = \underset{\hat{x}}{\operatorname{argmin}} E[\, ||\hat{x} - x||_\infty | y]$$

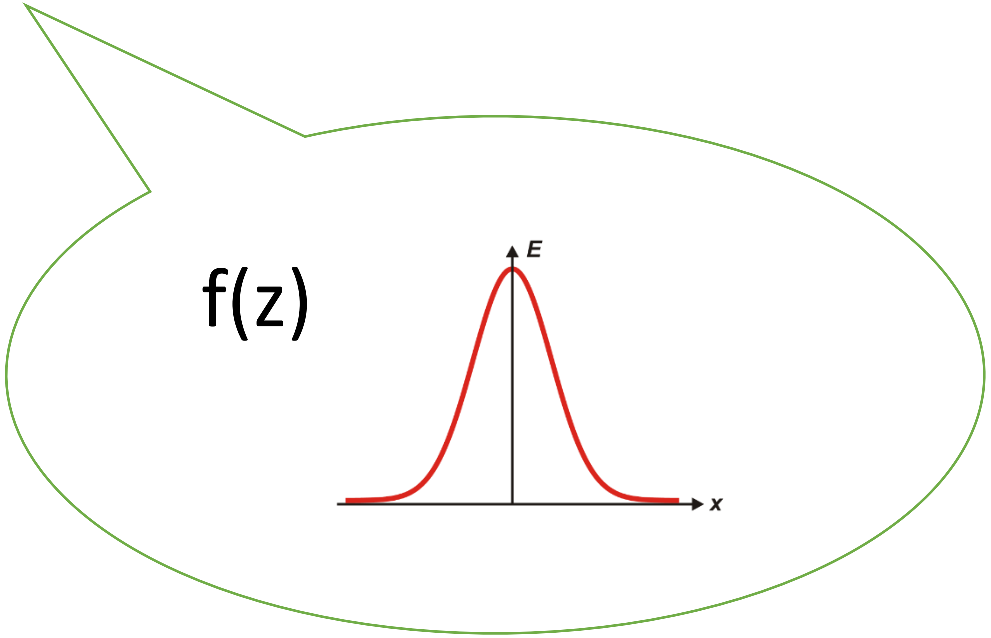
$f(x|y)$

Scalar channels

$$y = x + z$$



Observations

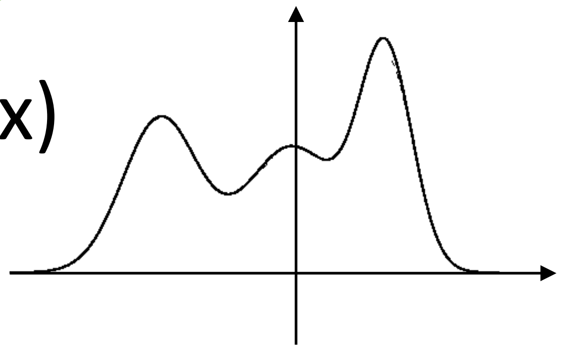


Noise

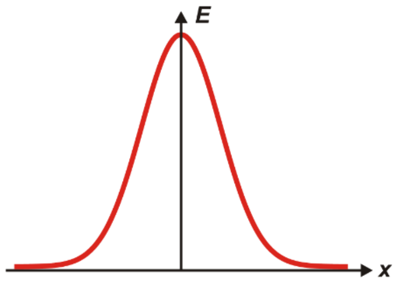
Scalar channels

$$y = x + z$$

f(x)



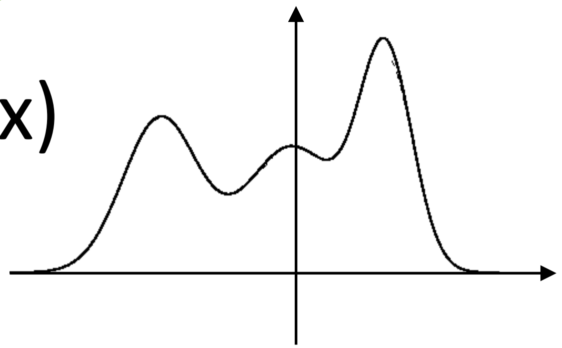
f(z)



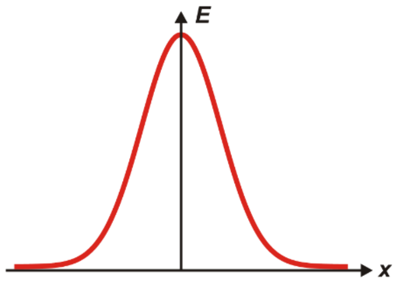
Scalar channels

$$y = x + z$$

f(x)



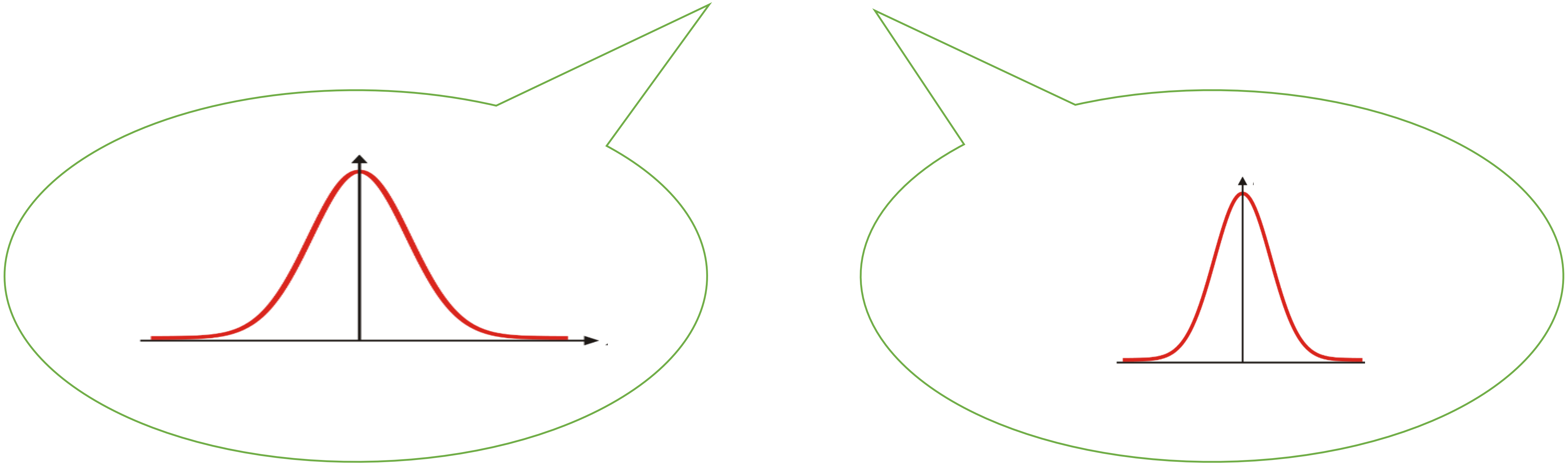
f(z)



✓ f(x|y) using Bayes' rule

Gaussian input

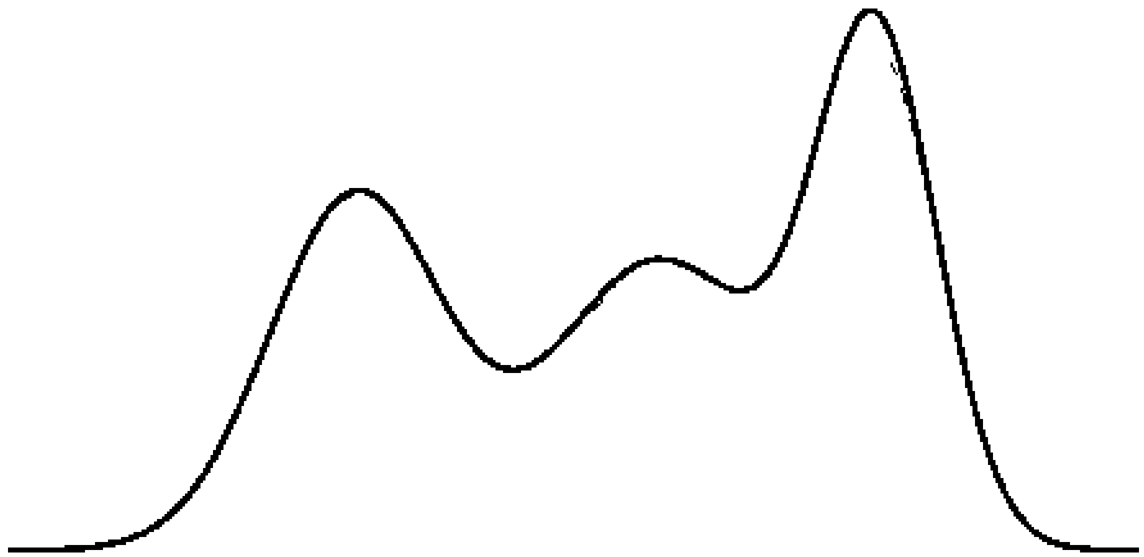
$$y = x + z$$



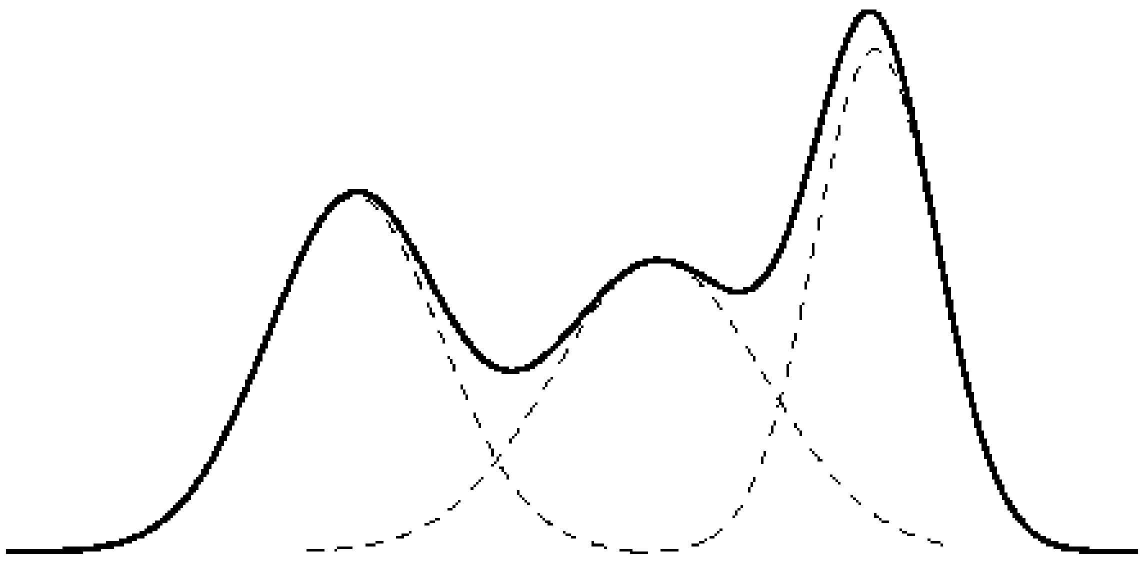
Wiener filter: $\hat{x} = c \cdot y$, where $c = \dfrac{\sigma_x^2}{\sigma_x^2 + \sigma_z^2}$

Optimal for ℓ_p errors, $p \geq 1$ [Sherman'58]

Input distribution

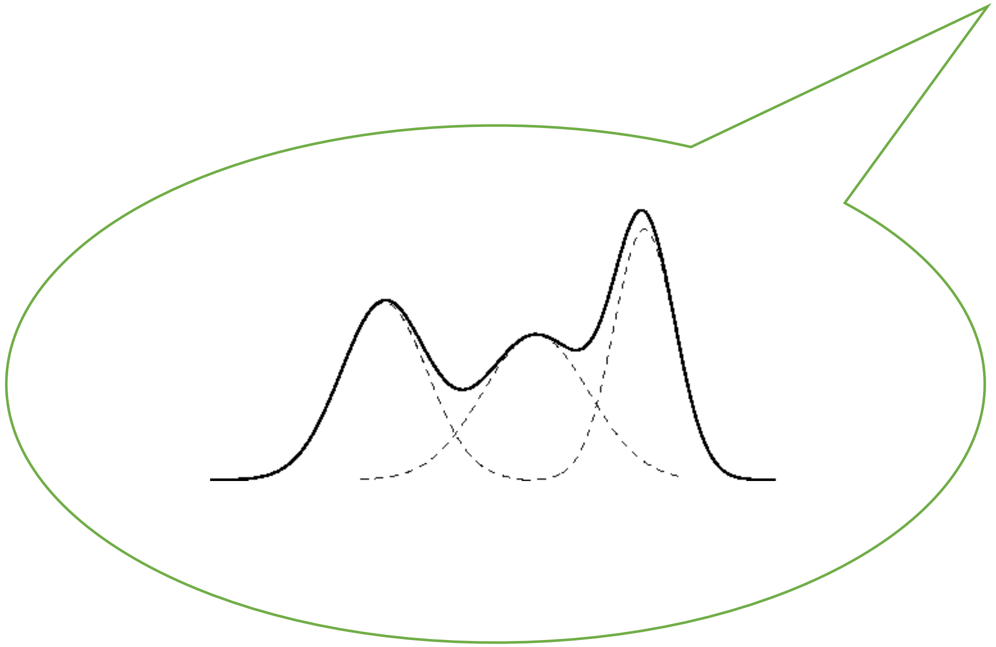


Gaussian mixture model [Alecu'06]



Gaussian mixture input

$$y = x + z$$

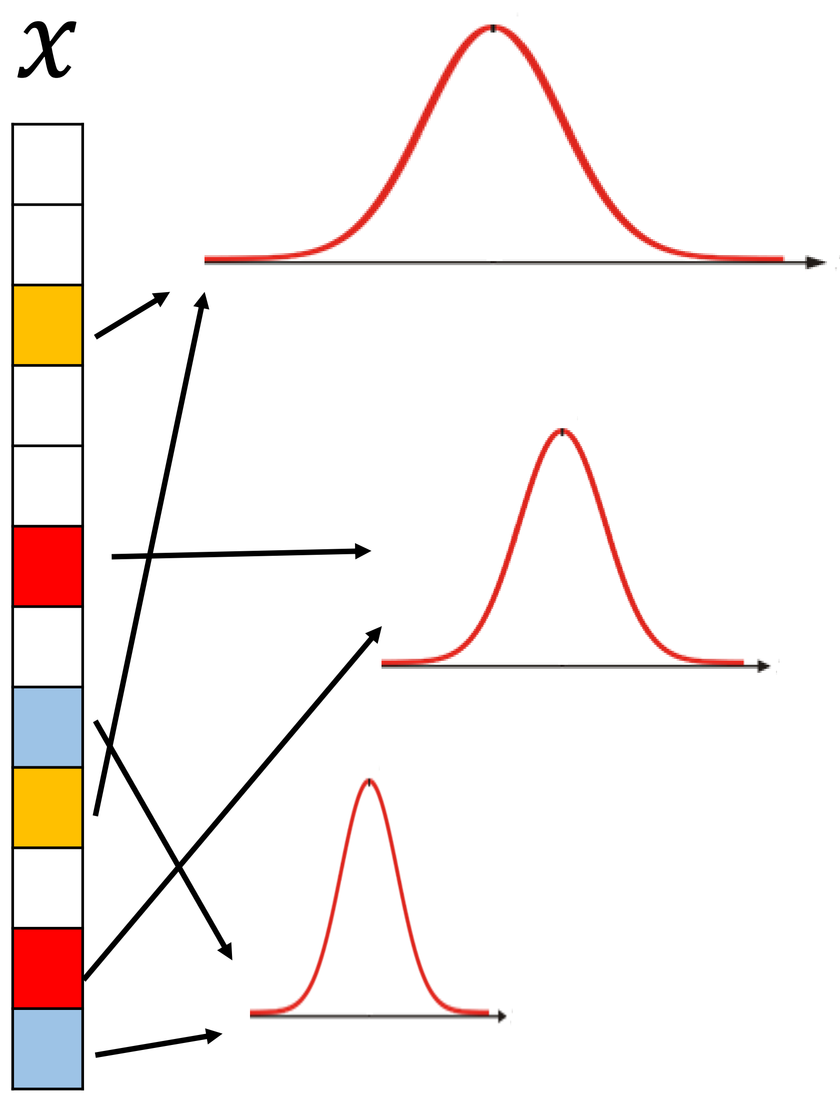


Multiple Wiener filters:

$$\hat{x}_1 = c_1 \cdot y$$
$$\hat{x}_2 = c_2 \cdot y$$
$$\hat{x}_3 = c_3 \cdot y$$

$$c_1 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_z^2} \qquad c_2 = \frac{\sigma_2^2}{\sigma_2^2 + \sigma_z^2} \qquad c_3 = \frac{\sigma_3^2}{\sigma_3^2 + \sigma_z^2}$$

Multiple Wiener filters:

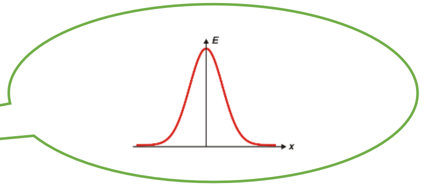


$$\hat{x}_1 = c_1 \cdot y$$

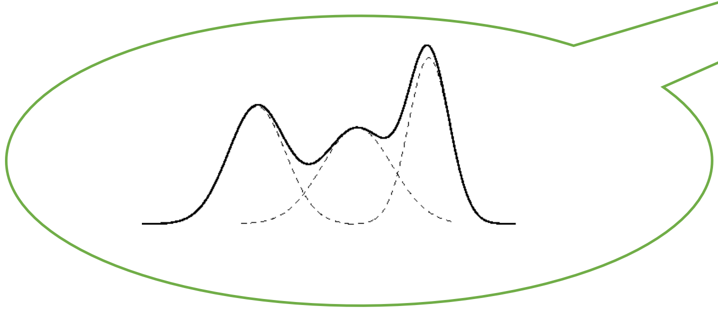
$$\hat{x}_2 = c_2 \cdot y$$

$$\hat{x}_3 = c_3 \cdot y$$

Theorem [Tan, Baron, and Dai '14]



$$y = x + z$$



Multiple Wiener filters:

$\checkmark$ $\hat{x}_1 = c_1 \cdot y$

$\hat{x}_2 = c_2 \cdot y$

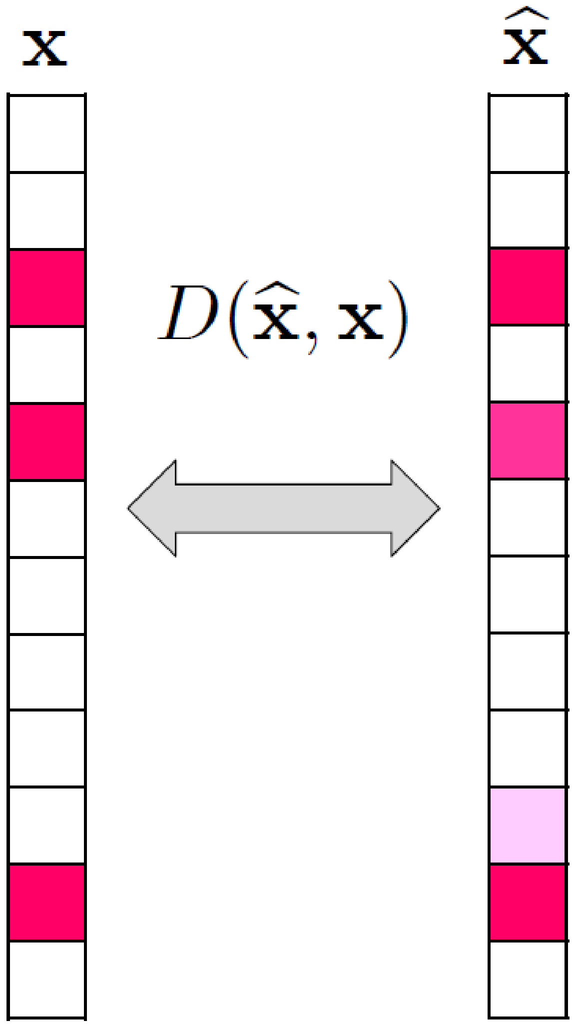
$\hat{x}_3 = c_3 \cdot y$

$$c_1 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_z^2}, \text{ where } \sigma_1^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$$

Limitation of Wiener filter

- Wiener filter $\hat{x}_1 = c_1 \cdot y$ minimizes ℓ_∞-norm error when $N \to \infty$.

- *What happens if N is finite?*

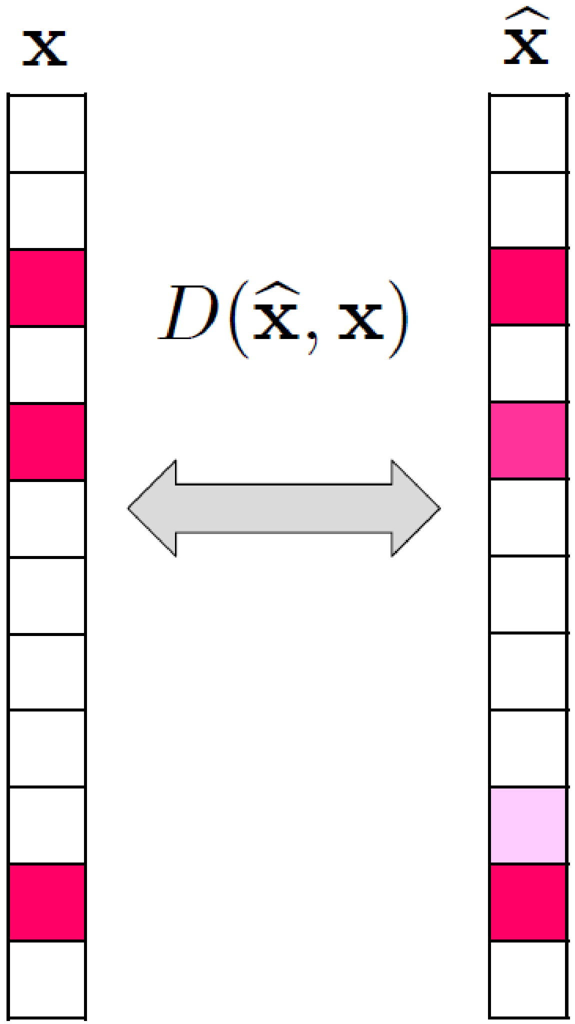


$$||\hat{x} - x||_\infty = \lim_{p \to \infty} ||\hat{x} - x||_p$$

$$D(\hat{x}, x) = ||\hat{x} - x||_\infty$$

$$D(\hat{x}, x) = \sum_i |\hat{x}_i - x_i|^p$$

$$\hat{x}_i = \operatorname{argmin} E[\,|\hat{x}_i - x_i|^p \,|y_i]$$



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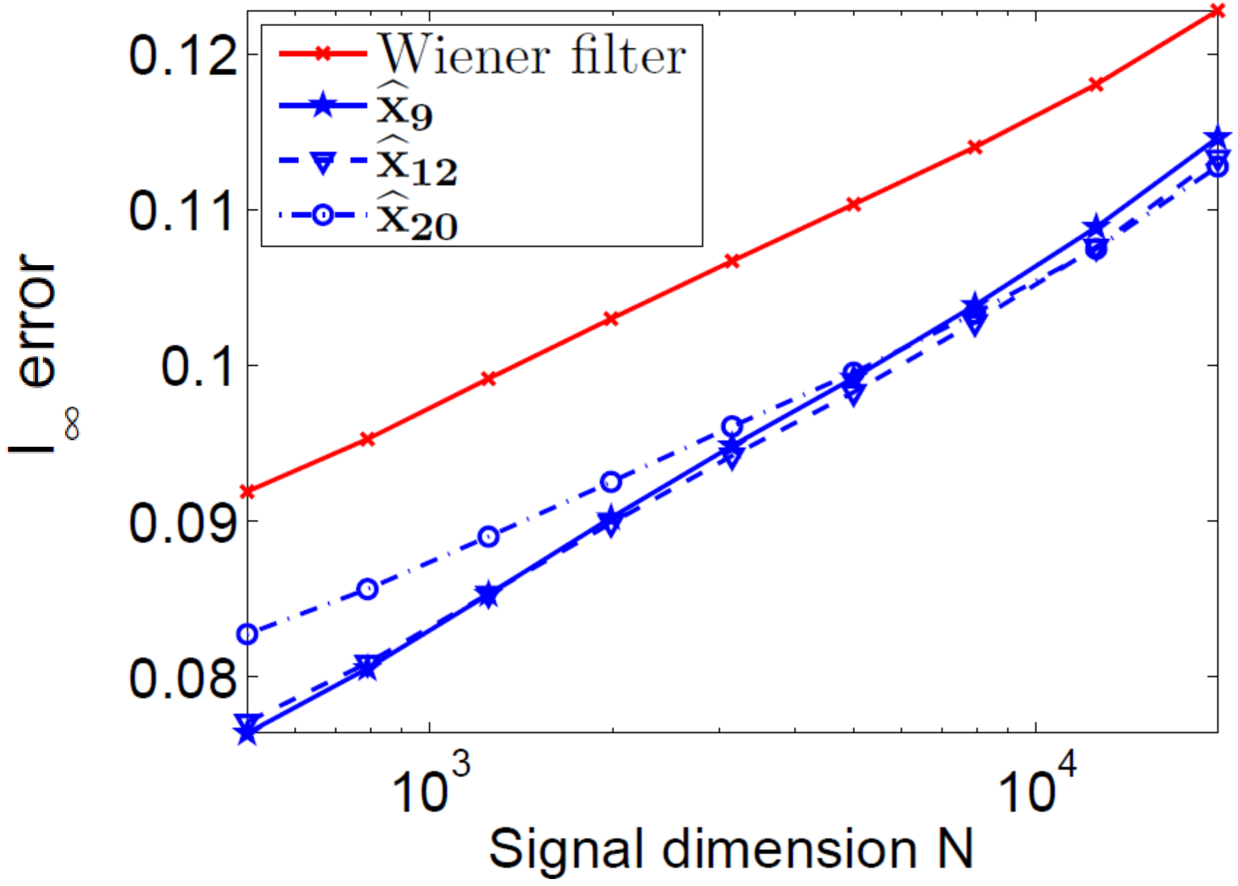
$$D(\hat{x}, x) = \sum_i |\hat{x}_i - x_i|^p$$

$$\hat{x}_i = \operatorname{argmin} E[\,|\hat{x}_i - x_i|^p\,|y_i]$$

Component-wise estimation!

Numerical results



$\hat{x}_9$ is estimated by

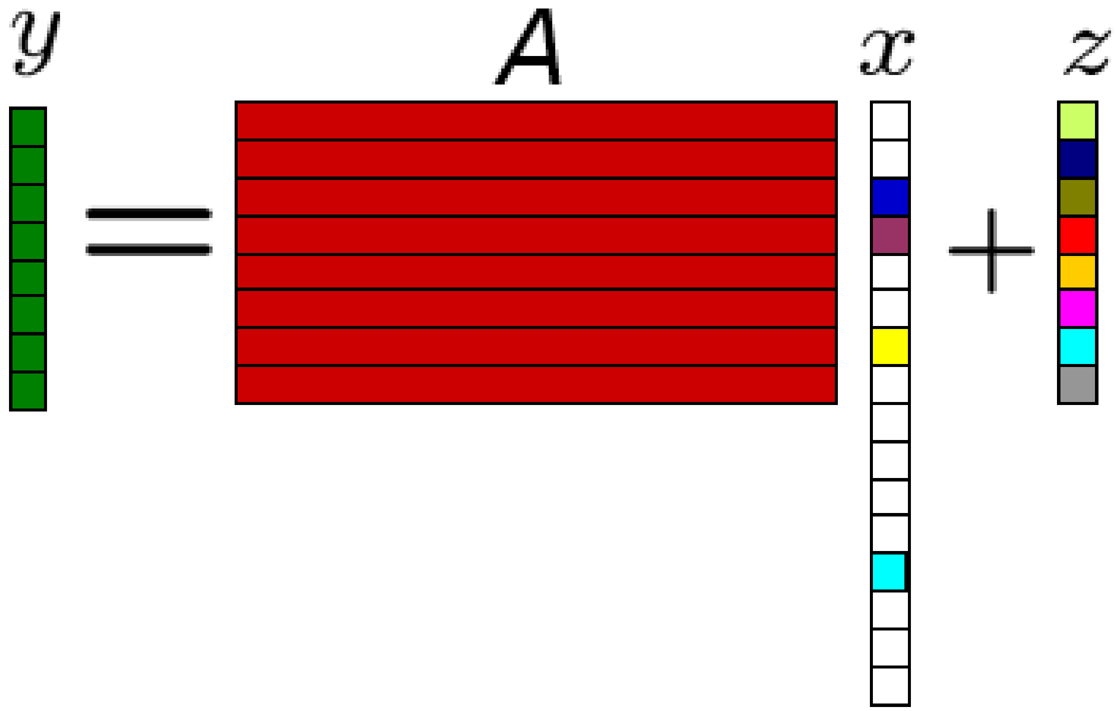
$$D(\hat{x}, x) = \sum_i |\hat{x}_i - x_i|^9$$

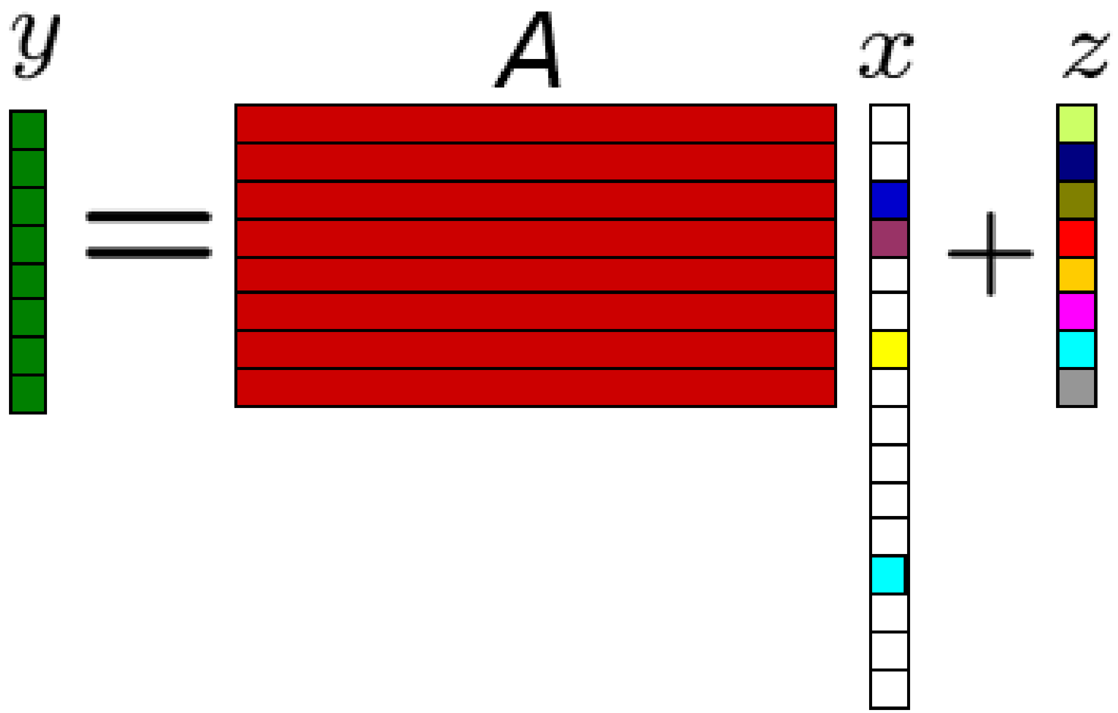
$$x \sim 0.01 \times N(0, 2) + 0.03 \times N(0,1) + 0.06 \times N(0,0.5) + 0.9 \times \delta(x)$$

Matrix channels

Matrix channels

$$y = Ax + z$$



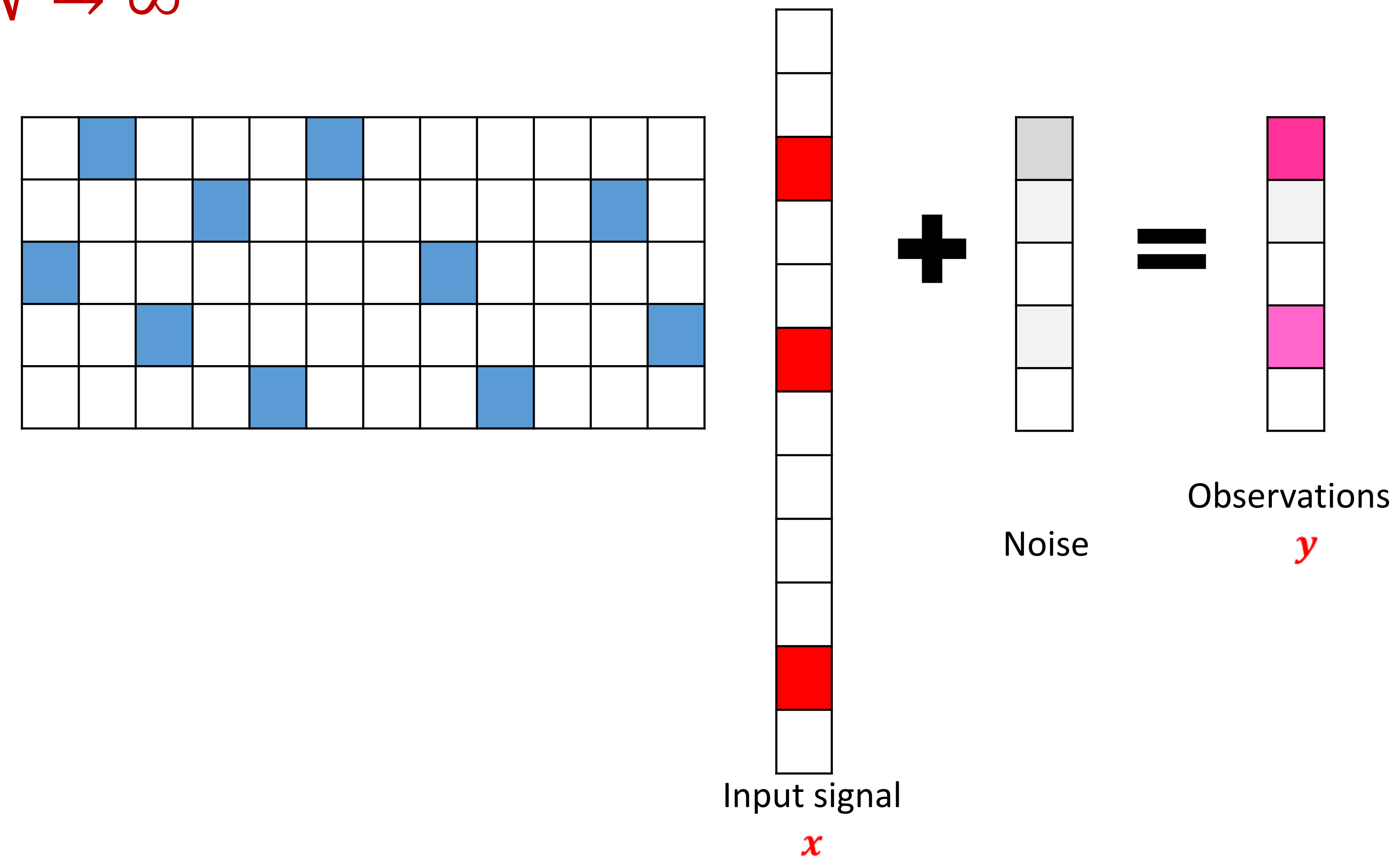


$$\hat{x}_\infty = \underset{\hat{x}}{\operatorname{argmin}}\, E[||\hat{x} - x||_\infty | y]$$

$f(x|y)$?

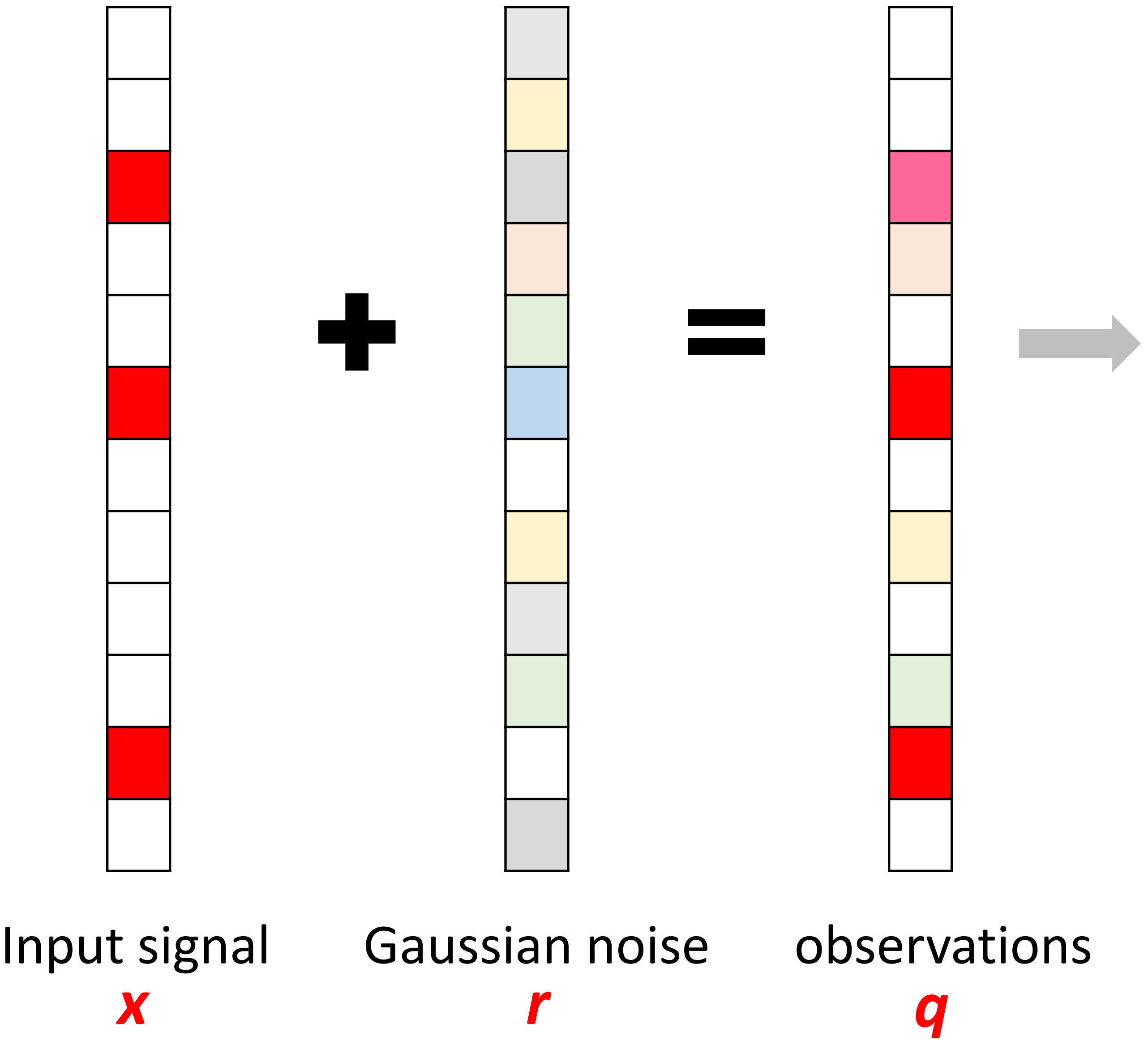
Decoupling principle [Tanaka '02, Guo & Verdu '05]

$N \to \infty$



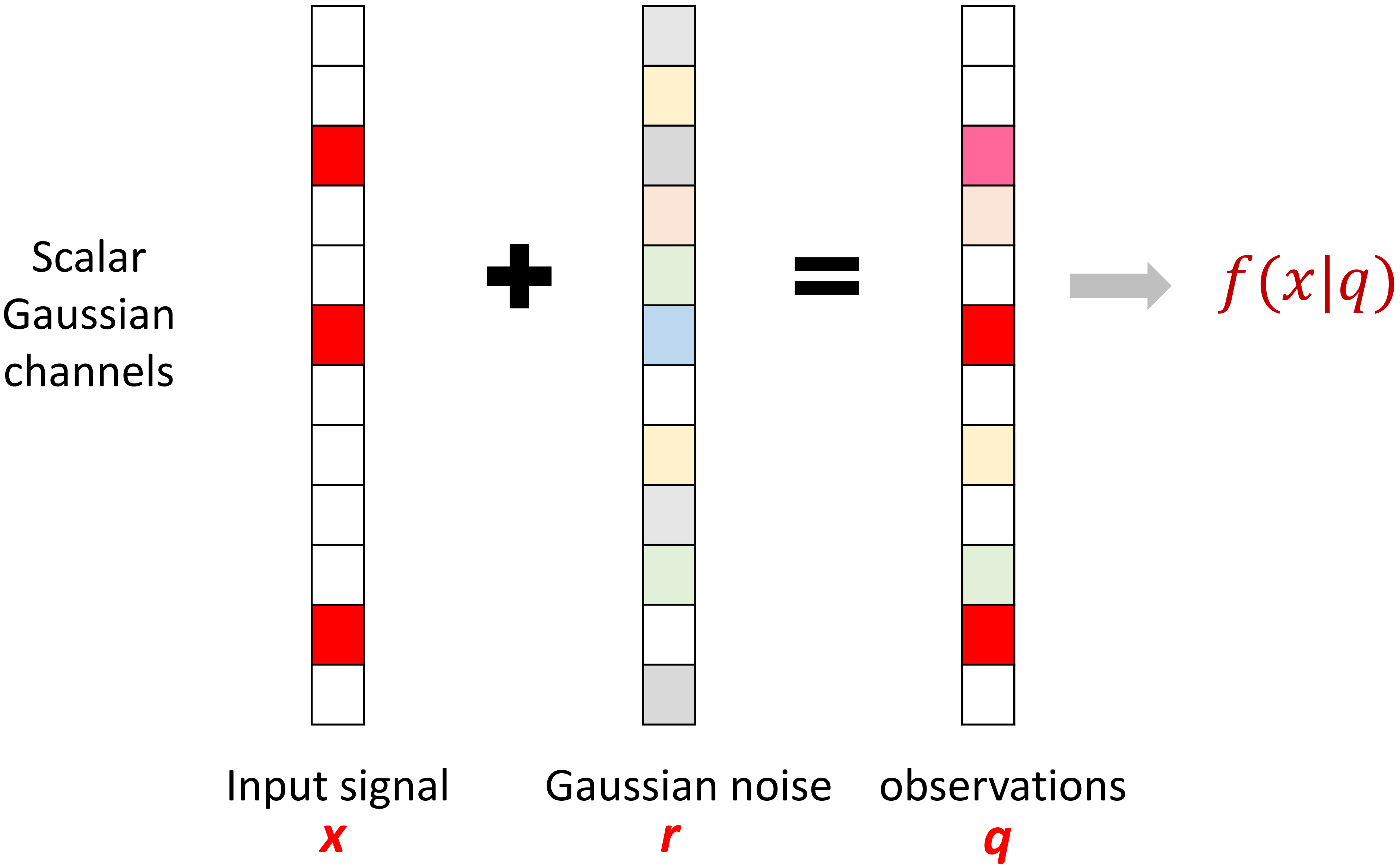
Decoupling principle [Tanaka '02, Guo & Verdu '05]

Scalar Gaussian channels

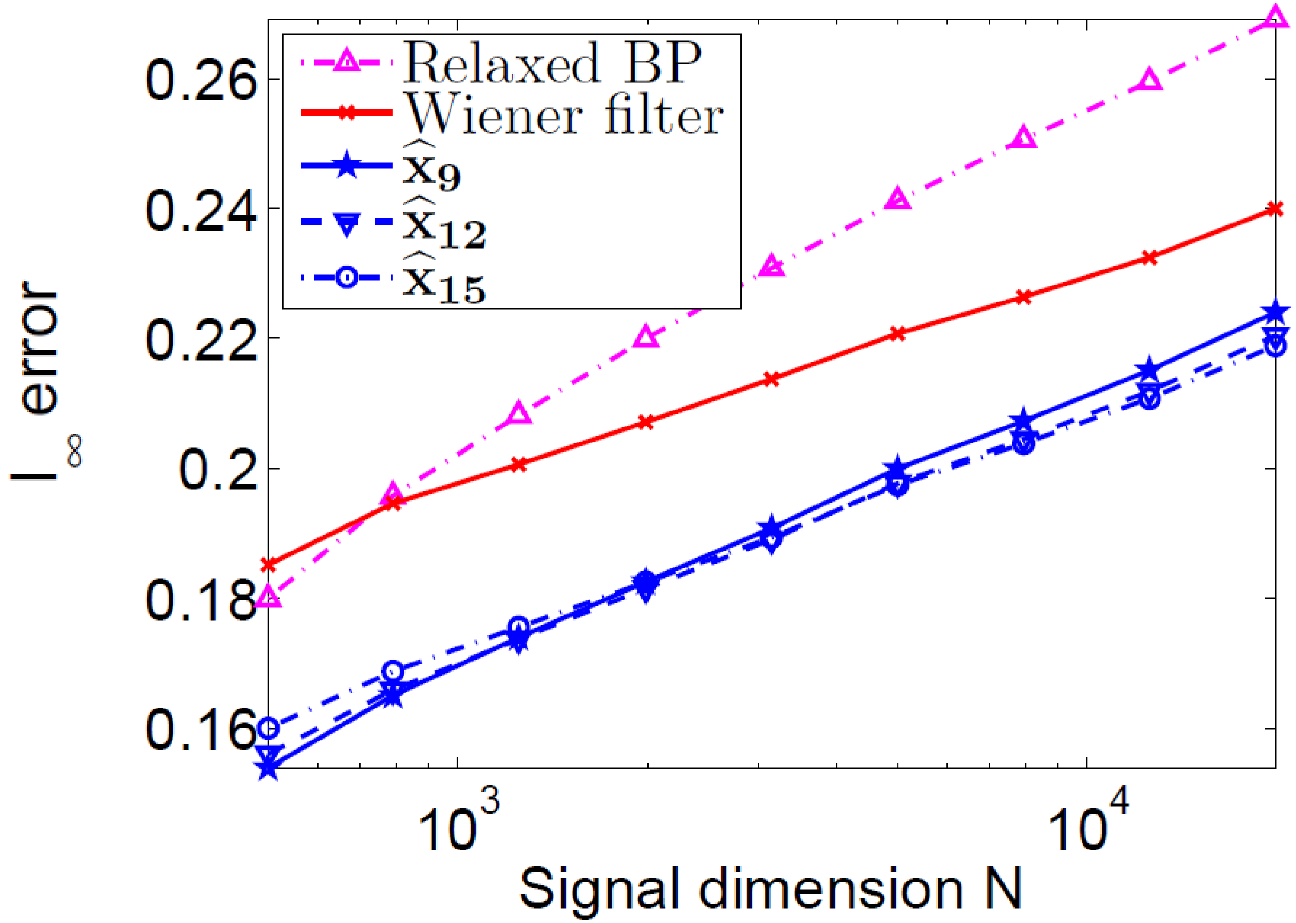


$f(x|q)$

Relaxed belief propagation [Rangan'10]



Numerical results

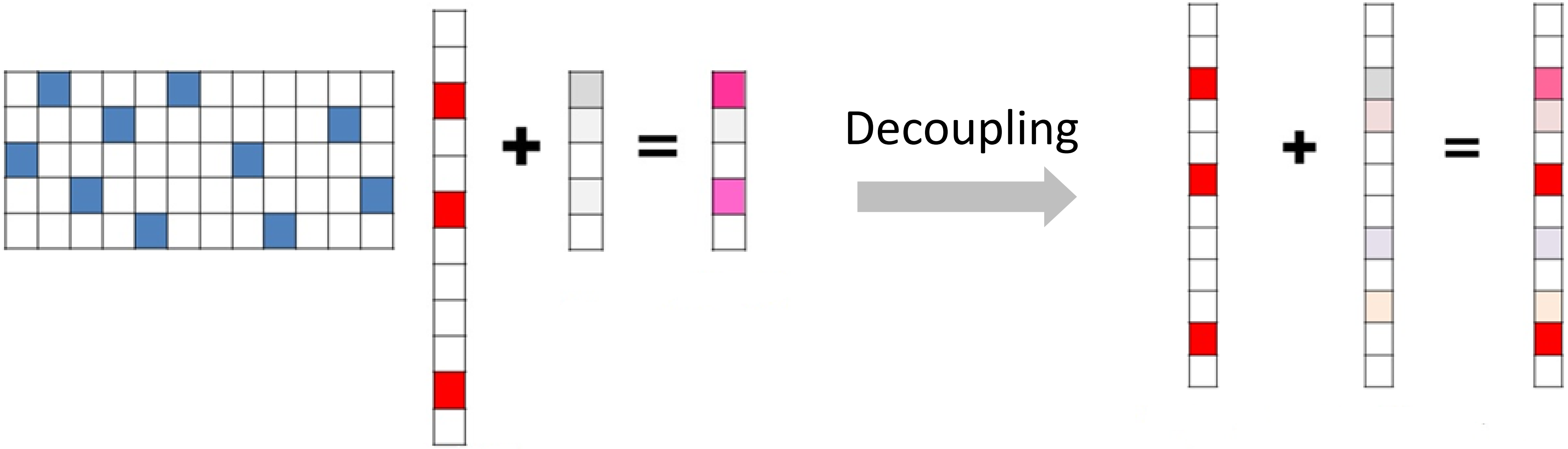


$\hat{x}_9$ is estimated by
$D(\hat{x}, x) = \sum_i |\hat{x}_i - x_i|^9$

$x \sim 0.01 \times N(0,2) + 0.03 \times N(0,1) + 0.06 \times N(0,0.5) + 0.9 \times \delta(x)$

Summary

- ℓ_∞-norm error
- Gaussian mixture input
- Wiener filter $\hat{x}_1 = c_1 \cdot y$
- $||\hat{x} - x||_\infty = \lim_{p\to\infty} ||\hat{x} - x||_p$



Thank you!