

ECE 301 (Section 003) Homework 3
Fall 2026, Dr. Dror Baron
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Maximum: 180 + Bonus 50

Problem 1 (20 pts) (Time-Invariance & Linearity Properties) Examine the time-invariance and linearity properties of the systems specified in Problem 7 of the previous HW.

Problem 2 (20 pts) (**Bonus 5 pts for part d**) (Time-Invariance & Linearity Properties) Examine the time-invariance and linearity properties of the systems specified in Problem 8 of the previous HW.

Problem 3 (20 pts) (Discrete Convolution Basics) Given signal $w[0] = -1$, $w[1] = 2$, $w[2] = -1$, and $w[n] = 0$, elsewhere, and signal $v[0] = 1$, $v[1] = 2$, $v[2] = 3$, $v[3] = 4$, and $v[n] = 0$, elsewhere.

- a) Express $w[n]$ and $v[n]$ in the self-referenced form. Plot them in two separate figures. Remember to label the axes.
- b) Decompose input signal $x[n] = w[n]$ into three subsignals, $x_i[n]$ for $i = 1, 2, 3$, send them into an LTI system with impulse response $h[n] = v[n]$, and calculate outputs $y_i[n]$ for $i = 1, 2, 3$ and subsequently $y[n]$ using the graphical approach. Clearly define your choices of $x_i[n]$ signals. Explain how $y_i[n]$ signals are obtained.
- c) Recalculate the output $y[n]$ by using the definition of convolution.
- d) Repeat part b) by letting $x[n] = v[n]$ and $h[n] = w[n]$. Are we getting the same result as in b)?

Problem 4 (20 pts) (Discrete Convolution)

- a) Use the definition of convolution, prove that for two generic signals $x[n]$ and $h[n]$, $x[n] * h[n] = h[n] * x[n]$. (You need to clearly demonstrate how a change of variables is done, and how the summation interval of the new dummy variable can be obtained from that of the old dummy variable.) How can this mathematical result be used to explain the finding in Problem 3d?
- b) Compute the convolution $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$. Consider the cases $\alpha \neq \beta$ and $\alpha = \beta$.
- c) Compute the convolution $y[n] = x[n] * h[n]$ for $x[n] = \left(-\frac{1}{2}\right)^n u[n-1]$ and $h[n] = 3^n u[1-n]$.
- d) Let $h[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$ be the impulse response of a discrete-time LTI system with input $x[n]$ given by:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 2, \\ 0, & \text{elsewhere.} \end{cases} \quad (1)$$

- (i) Find the output $y[n]$ of the system.
- (ii) Is $y[n]$ periodic? If yes, find its fundamental period.

Problem 5 (20 pts) (The Human Body as an “LTI System”) The concentration of an injected (or orally ingested) drug in the blood is commonly modeled as a decaying exponential. This means that if a human body is injected with a drug to time 0, its concentration after n hours is given by $h[n] = \alpha^n u[n]$, where α is defined by the half-life $n_0 > 0$, meaning that $h[n_0] = 0.5h[0]$. You can think of a drug injection as an impulse $x_0[n] = \delta[n]$ and $h[n]$ as the impulse response of the human body, which metabolizes the drug over time.

- a) If the half-life of a certain drug is 8 hours, what is $h[n]$? Also, find $y[n] = h[n] * x_0[n]$.
- b) If instead of an injection, the drug is administered through a steady intravenous (IV) drip starting at time 0, what will be the concentration of the drug as a function of time? Specifically, if $x_1[n] = 0.2u[n]$ find $y[n] = x_1[n] * h[n]$.
- c) The drip procedure is not producing a sufficient response. So, a strong extra dose of the drug is injected after 8 hours into the dripping process, namely $x_2[n] = 2\delta[n - 8]$. Find the new concentration $y[n]$ as a function of time.
- d) Compare the concentrations of the drug in the blood after 16 hours under the three different scenarios, (i) injection at time 0, (ii) drip starting at time 0, (iii) drip starting at time 0 and then injection as modeled by $x_2[n]$.

Problem 6 (20 pts) (Applications of Convolution in Networking and Communications)

- a) A lossless computer network with a reflection is found to have impulse response

$$h(t) = \delta(t - 50 \text{ nsec}) - 0.7\delta(t - 100 \text{ nsec}). \quad (2)$$

A rectangular pulse $x(t) = u(t) - u(t - T)$ is input into the network and $y(t) = h(t) * x(t)$ is received. Plot $y(t)$ over $0 \leq t \leq 0.3 \mu\text{sec}$ if the pulse length T is given by $T = 70 \text{ nsec}$.

- b) A transmitter $s(t)$ inputs a signal into $c(t)$, a wireless channel having reflections. The channel has the following impulse response:

$$c(t) = A_d\delta(t - t_d) + A_1\delta(t - t_d - t_1), \quad (3)$$

where $A_d = 1$, $t_d = 16.67 \text{ nsec}$, $A_1 = 0.95$, and $t_1 = 2 \text{ nsec}$. The sinusoidal carrier frequency is given by $f_c = 0.5 \text{ GHz}$. The transmitted signal is

$$s(t) = \begin{cases} \sin(2\pi f_c t), & 0 \leq t \leq 3T_p, \\ 0, & \text{else,} \end{cases} \quad (4)$$

where T_p is one period of the carrier. (i) Compute and plot the impulse response $c(t)$ of this channel. (ii) The received signal $r(t)$ is given by the convolution $r(t) = c(t) * s(t)$. Plot $r(t)$ over $0 \leq t \leq 50 \text{ nsec}$. [Note: Use software to plot $r(t)$.]

Problem 7 (20 pts) (Evaluate Convolution) Perform convolutions of the following functions. Use hand drawings to determine the intervals. You will need to show intermediate calculation steps to get full points.

- a) (6 pts) $x(t) = \text{rect}(t)$ and $h(t) = \text{rect}(2t - 1/2)$.
- b) (6 pts) $x(t) = e^{-2t}u(t)$ and $h(t) = e^{-3t}u(t)$.
- c) (8 pts) $x(t) = e^4[u(t+3) - u(t)]$, $h(t) = e^{-3t}[u(t-1) - u(t-2)]$.

Problem 8 (20 pts) (Implement Convolution) Use Matlab to implement the convolution between

$$x[n] = 0.5^n \cos(n\pi/4)(u[n] - u[n-10]), \text{ and}$$

$$h[n] = 0.6^n \sin(n\pi/3)(u[n] - u[n-10]).$$

Plot the input, output, and the impulse response on the same graph (2 pts). Use different colors for different signals (2 pts). (Type “`help plot`” to see how to do it.) Use `legend()` to create a descriptive label for each plotted signal (2 pts). Properly label the axes (2 pts). Append your source code to the submission (12 pts). Note that you need to implement the convolution operation by yourself, instead of using the built-in function `conv()` from Matlab. You can use `conv()` to verify the correctness of your implementation. Inserting zeros to the beginning or the end of the vector x and/or vector h may help you avoid negative indexing issues.

Problem 9 (20 pts) (Properties of LTI Systems) Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

- a) $h(t) = e^{-2t}u(t) + e^{t/100}u(t-1)$
- b) $h[n] = (n+1)2^n u[n]$
- c) $h[n] = 4^n u[2-n]$
- d) $h(t) = e^{-2t}[u(t) - u(t-1)]$.

Bonus Problem 1 (5 pts)

Consider the following integral:

$$y(t) = \int_0^t x(\tau) d\tau, \quad t \geq 0,$$

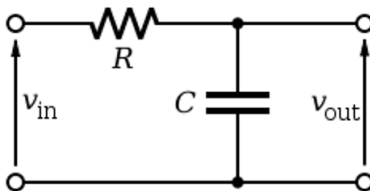
where $x(t)$ is a piecewise function defined below:

$$x(t) = \begin{cases} \log_{10} 1000t^2, & 0 \leq t < 2, \\ 4 - t, & 2 \leq t \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $y(t)$ for all $t \in \mathbb{R}$.

Bonus Problem 2 (20 pts) (Delta Function as a Limiting Function)

- a) (15') The RC circuit below has supply voltage $v_{\text{in}}(t)$ and capacitor voltage $v_{\text{out}}(t)$. We consider the supply voltage as input of the system, i.e., $x(t) = v_{\text{in}}(t)$ and capacitor voltage as the output, i.e., $y(t) = v_{\text{out}}(t)$. The output is $y(t) = [1 - e^{-5(t-t_0)}] u(t - t_0)$ when supplied with a unit-step input voltage $x(t) = u(t - t_0)$.



- (i) Without going into mathematical derivation/manipulation, argue with intuition why the RC circuit can be viewed as a time-invariant system.
- (ii) Suppose the input voltage is 1 during the interval of $t = 0$ to 0.5 seconds and 0 otherwise, i.e., $x(t) = u(t) - u(t - 0.5)$. Using the superposition principle you learned from the circuit theory (note that it is equivalent to the superposition property in this course), show that the analytic form of the output voltage is

$$y(t) = \begin{cases} 1 - e^{-5t}, & 0 \leq t \leq 0.5, \\ 11.18e^{-5t}, & t > 0.5, \\ 0, & \text{else.} \end{cases}$$

- (iii) How is the superposition principle related to the linearity property of the system of the RC circuit?
 - (iv) What are the analytic forms of the output voltage given the following input voltage?
 - (1) $x(t) = 2[u(t) - u(t - 0.5)]$
 - (2) $x(t) = 4[u(t) - u(t - 0.25)]$
 - (3) $x(t) = 20[u(t) - u(t - 0.05)]$
 - (v) Using the same amplitude scale and time scale on the same Matlab plot, visualize the output voltages $y(t)$ obtained in (iv) over $0 \leq t \leq 1$. Note that, using (iv)(1) as an example, you can either draw a piecewise function separately for $t = 0 : 0.01 : 0.5$ and $t = 0.5 : 0.01 : 1$ or in one piece for $t = 0 : 0.01 : 1$.
- b) (5') You should notice that the sequence of curves $y(t)$ you plotted for a)(iv) are converging. Mathematically prove that the limiting curve is the delta function. [Hint: Guess a common expression for all $x(t)$'s in a)(iv) and parameterize the expression by a variable. Drive the variable to an extreme value and see the impact to $y(t)$.]

Bonus Problem 3 (20 pts) (Associative Property of Convolution)

- a) (4') Prove the following associative property of convolution from the definition:

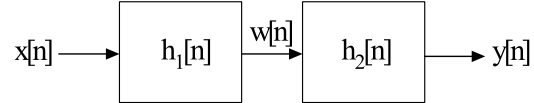
$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)].$$

Note that you can only earn points by following the definitions and doing changes of variables correctly.

b) Consider two LTI systems with the impulse responses

$$h_1[n] = \left(-\frac{1}{2}\right)^n u[n],$$
$$h_2[n] = u[n] + \frac{1}{3}u[n-1].$$

These two systems are cascaded as shown in the following figure. Let $x[n] = u[n]$.



- i) (6') Prove that $u[n] * u[n] = (n+1)u[n]$ and $u[n] * u[n-1] = nu[n-1]$.
- ii) (10') Verify the associative property of convolution by showing that $y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$. **[Hint:** Try to apply the distributive property of convolution for the most parts of the proof. At the very last step, apply the definition of convolution or use the graphical approach to obtain the final solution.)