

ECE 301 (Section 003) Homework 4
Fall 2026, Dr. Dror Baron
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Maximum: 100 + Bonus 5

Problem 1 (20 pts) (Vector and Matrix Refresh) Seven data points are arranged as columns of a data matrix $\mathbf{X}$ given as follows:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 & -2 \\ 2 & 0 & 1 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

- a) Draw all data points on a 2D plane by hand. Properly label the two axes. Clearly mark important axis tick values to facilitate a precise graphical description of the data.
- b) Consider each point as a vector. Calculate the angle (in $^\circ$) between $[2 \ 2]^T$ and the five other points (excluding $[0 \ 0]^T$), respectively, using the inner/dot product formula that involves the angle. Note that the angle between two vectors can be negative.
- c) Calculate the matrix outer product for $\mathbf{X}$, namely, $\mathbf{R} = \mathbf{X}\mathbf{X}^T$. Show the intermediate steps of calculating each element of the 2-by-2 matrix $\mathbf{R}$.
- d) The matrix outer product can also be calculated via $\mathbf{R} = \sum_{i=1}^7 \mathbf{x}_i \mathbf{x}_i^T$, where $\mathbf{x}_i$ is the i th column of $\mathbf{X}$. Evaluate the numerical result. Note that each $\mathbf{x}_i \mathbf{x}_i^T$ is a 2-by-2 matrix.
- e) Now, consider each column of $\mathbf{X}$ as a *single* block-entry of the matrix. Rewrite $\mathbf{X}$ and $\mathbf{X}^T$ into the form of the vector of blocks-entries, respectively. Use the vector multiplication rule to show that $\mathbf{R} = \sum_{i=1}^7 \mathbf{x}_i \mathbf{x}_i^T$.

Problem 2 (20 pts) (**Bonus 5 pts for part c**) (Linearly independence, Basis, and Vector Space))

- a) Are vectors $[1 \ 2]$, $[4 \ 5]$, and $[7 \ 8]$ linearly independent? What about $[1 \ 2 \ 0]$, $[1 \ -1 \ 1]$, and $[0 \ 0 \ 1]$? Justify your answers.
- b) You are given a vector space $V = \text{span}\{[-1 \ 0 \ 0], [0 \ -1 \ 0]\}$.
 - (i) Express V in a set representation.
 - (ii) Can you find a basis for V ?
 - (iii) Are $[5 \ 8 \ 0]$, $[8 \ 0 \ 5]$, and $[0 \ 5 \ 8]$ in vector space V ? If yes, what are the coefficient for each vector of the basis you found in (ii)?
 - (iv) Draw all points of (iii) in a 3D coordinate. Illustrate vector space V using a plane formed by the vectors of the basis.
- c) (**Bonus, 5 pts**) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -3 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

What is the dimension of the column vector space of $\mathbf{A}$? What is the rank of $\mathbf{A}$?

Problem 3 (20 pts) (Orthogonal and Unitary Matrices)

You have learned in class that a real-valued *orthogonal matrix* $\mathbf{P}$ has the property that $\mathbf{P}^T \mathbf{P} = \mathbf{P} \mathbf{P}^T = \mathbf{I}$. In signal processing, oftentimes signals are represented in complex values. We can define for a complex-valued square matrix a similar concept called the *unitary matrix*. A unitary matrix $\mathbf{Q}$ satisfies the property that $\mathbf{Q}^H \mathbf{Q} = \mathbf{Q} \mathbf{Q}^H = \mathbf{I}$, where “ H ” is the Hermitian operator that is a combination of both the transpose, “ T ,” and the complex conjugate, “ $*$.”

- a) The discrete Fourier transform (DFT) matrix for a signal of length 3 is defined as $\mathbf{Q}_3 = [q_{kn}]_{k,n \in \{0,1,2\}}$, where $q_{kn} = \frac{1}{\sqrt{3}} \exp(-j \frac{2\pi}{3} kn)$. Explicitly write out this 3-by-3 matrix. Simplify each entry but do not evaluate numerically the complex exponentials.
- b) Verify that $\mathbf{Q}_3$ is a unitary matrix.
- c) Compute the forward discrete Fourier transform for signal $\mathbf{x} = [1.01, 0.99, 0.97]^T$ to obtain a transformed signal $\mathbf{z} = \mathbf{Q}_3 \mathbf{x}$, where $\mathbf{z} = [z_0, z_1, z_2]^T$. How large are the norms for z_0 , z_1 , and z_2 ?
- d) Now, zero out z_2 to obtain a new vector $\mathbf{z}_{\text{compressed}}$. Compute the inverse transform using $\hat{\mathbf{x}} = \mathbf{Q}_3^H \mathbf{z}_{\text{compressed}}$. Is $\hat{\mathbf{x}}$ similar to $\mathbf{x}$? Can you guess why?

Problem 4 (20 pts) (Projection and PCA) You are given a data matrix

$$\mathbf{X} = \begin{bmatrix} -0.92 & 1.09 & -1.35 & 2.06 & -0.60 & -0.28 \\ -0.08 & 1.15 & -1.67 & 1.08 & -1.14 & 0.66 \end{bmatrix}.$$

and its principal component $\mathbf{u} = [0.7474, 0.6644]^T$.

- a) Plot the data points using circles and the principal component using a line segment with one end at $[0, 0]^T$ on a 2D plane using Matlab.
- b) Calculate the projection c_i for each data point $\mathbf{x}_i$ to $\mathbf{u}$. Note that c_i can be negative.
- c) Calculate the sample variance of $\{c_i\}_{i=1}^n$, namely,

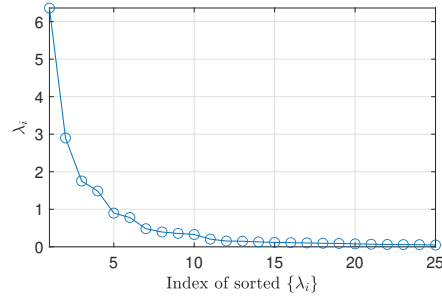
$$\widehat{\text{Var}}(\{c_i\}) = \frac{1}{n-1} \sum_{i=1}^n (c_i - \bar{c})^2,$$

where $\bar{c} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n c_i$ is the sample mean.

- d) Repeat b) and c) using another unit vector $\mathbf{u}_a = [0.8, 0.6]^T$. Is the newer sample variance smaller than that in b)? Can you explain why?

Problem 5 (20 pts) (PCA on Downsampled Yale Face Database) In this problem, we will explore PCA as a visualization tool for Yale Face Database. Download the .m files and the database from this link and this link. Extract the face image files into a folder named `yalefaces` and put the .m files at the same level of the folder. Call this folder “problem_pca_face/”. Open Matlab, change your “Current Folder” to “problem_pca_face/”, and open `main_pca_visualization.m`.

- a) Run the code corresponding to this part only, describe the data structure of variable `img_buffer`. Set `preview_img_flag` to 1, re-run the code to visually inspect the whole database.
- b) Complete Matlab function `[V, Lambda_mat] = PcaViaKlt(data)` by implementing PCA using eigendecomposition on a sample covariance matrix of the face data. The detailed information about the input and outputs are given in the comments of the incomplete function. You may use built-in function `eig` for eigendecomposition. If your implementation is correct, after running the code of b), you will obtain a plot similar to the following.



- c) Run the code of c) to visualize a few dominating eigenvectors. Comment on whether they reflect some characteristics of the faces you saw in a).
- d) The code of d) projects each face image (coming from one of the four selected classes) onto a 2D space. Comment on PCA's data visualization performance in this specific example.