ECE 592–100 – Signal Processing Tour of Quantum Computing
Homework 4

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Due date – February 21, 2024

Administrative instructions:

1. The homework must be submitted individually or in pairs.
2. You should submit electronically on Moodle by midnight the day that the homework is due.
3. Scans of handwritten answers are fine.
4. Please justify your answers carefully.

Question 1 (Period of signal.)
Consider the discrete time signal, $x(n) = \sin(20\pi n) + \cos(30\pi n)$. What is the period of this signal?

Question 2 (Nyquist rate and sampling.)
Consider the discrete time signal, $x_a(t) = \cos(50\pi t)$. What is the Nyquist rate, F_n (what is the minimum number of samples per unit time required)?
Programming component. Select some sampling rate F_s below the Nyquist rate. Design another signal $x'_a(t)$ that would alias to the same discrete time outcome. Finally, demonstrate the aliasing effect visually by plotting $x_a(t)$, $x'_a(t)$, and how at multiples of $T = 1/F_s$ their values match.

Question 3 (Fourier transform.)
Consider a periodic discrete time signal with period $N = 4$, $x = \{\underline{1}\ 0\ 0\ 0\}$. (The underline corresponds to time index $n = 0$.) Compute the Fourier coefficients, C_k, for $k \in \{0, 1, 2, 3\}$.

Question 4 (Analysis of LTI system.)
Consider a simple discrete time system, H, whose input $x(n)$ and output $y(n)$ obey the following relation,

$$y(n) = 0.7y(n-1) + x(n).$$

(a) Compute the impulse response, $h(n)$.
(b) Compute the Fourier transform, $H(\omega)$ of $h(n)$. (Note that Question 3 considered Fourier of a periodic signal, while here $h(n)$ is aperiodic.)
(c) Consider an input, $x(n) = \sin(0.2\pi n)$. What will be the corresponding output, $y(n)$?
(Hint: one way to solve part (c) involves looking at the magnitude and phase of $H(\omega)$, and then plug them into the "sinusoids get mapped to sinusoids" expression we discussed in class. Another approach is to decompose $x(n)$ into the sum of 2 complex exponents, $x_1(n) = e^{-i0.2\pi n}$ and $x_2(n) = e^{+i0.2\pi n}$, compute the corresponding $y_1(n)$ and $y_2(n)$, and so on. Your output $y(n)$ should be real-valued.)

Question 5 (Sub-sampling a signal.)
Consider taking a discrete time signal $x(n)$ and evaluating only even numbered samples, $\widetilde{x}(n) = x(2n)$. We can compute

$$\widetilde{X}(\omega) = \sum_{n=-\infty}^{+\infty} x(2n)e^{-in\omega}$$

using values of $x(n)$. In this question, we will express $\widetilde{X}(\omega)$ using $X(\omega)$.
(a) Define $x_a(n) = x(n)(-1)^n$. Express $X_a(\omega)$ in terms of $X(\omega)$ without using the actual sample values, $x(n)$. (Hint: consider the form $X_a(\omega) = c_1 X(c_2\omega + c_3)$ for some constants.)
(b) Define $x_b(n) = x(n) + x_a(n)$. How does $x_b(n)$ relate to $\widetilde{x}(n)$?
(c) Express $\widetilde{X}(\omega)$ using the above results.

Question 6 (Low pass filters.)
The following signals can all be interpreted as low pass filters, meaning that they attenuate higher frequencies while letting lower ones pass through.
(a) Consider an averaging filter with $2M+1$ taps. That is, $h(n) = \frac{1}{2M+1}$ when $|n| \leq M$, else $h(n) = 0$. What is $H(\omega)$? (Note that the value $\frac{1}{2M+1}$ ensures that $|H(0)| = 1$.)
(b) Consider a triangular filter defined over the same range, $|n| \leq M$. Samples $h(n)$ have the form $h(n) = (M+1-|n|)/A$ for $n \in \{-M, -M+1, \ldots, M\}$. Note that $|h(0)|$ is largest. Design A in a way that the filter's DC response is 1, meaning that it barely impacts low frequencies.
Programming component. Compare the frequency response, $|H(\omega)|$, for these 2 filters. Discuss how the frequency responses compare.