ECE 592-100 – Signal Processing Tour of Quantum Computing

Final Exam – Spring 2023

May 1, 2023

Please remember to justify your answers carefully.

Last name: ____________________ First name: ____________________

Question 1 (Two-qubit quantum circuit.)
Consider a simple quantum circuit with two inputs,

$$|\phi_1\rangle = a_1 |0\rangle + b_1 |1\rangle ,$$

$$|\phi_2\rangle = a_2 |0\rangle + b_2 |1\rangle .$$

In this question, you will pass these qubits through two controlled NOT (CNOT) gates. You will analyze how the state vector of the two-qubit system evolves, and specify your state vectors as length-4 column vectors.
(a) The initial state is $|\psi_1\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$. Compute $|\psi_1\rangle$. (Recall that you must provide a length-4 vector.)
(b) The next state, $|\psi_2\rangle$, is obtained by applying a CNOT, where the first qubit ($|\phi_1\rangle$ in our case) is the control and the second qubit ($|\phi_2\rangle$) is the target. Compute $|\psi_2\rangle$.
(c) The last state, $|\psi_3\rangle$ is obtained by applying a CNOT in an opposite way, meaning that the first qubit is the target and the second qubit is the control. Compute $|\psi_3\rangle$.

Question 2 (Implementing Toffoli gates.)
A Toffoli gate can be thought of as a CNOT with two control qubits. Denoting the control qubits by c_1 and c_2 and the target qubit by t, a Toffolio gate takes the classical triple (c_1, c_2, t) to $(c_1, c_2, t \oplus c_1 c_2)$.
More generally, a quantum gate with two control qubits that applies the single qubit operation U to t when c_1 and c_2 are both 1 is denoted by $C^2(U)$. That is,

$$C^2(U) |c_1 c_2 t\rangle = |c_1 c_2\rangle U^{c_1 c_2} |t\rangle .$$

It is well known in the quantum computing literature (details in Exercise 4.21 and Figure 4.8 in Nielsen & Chuang) that a controlled-U gate can be implemented using several CNOT, V, and $V^\dagger$ gates, where $V^2 = U$. Nielsen & Chaung describe the V matrix needed to implement the Toffoli gate, for which $U = X$. Compute V such that $V^2 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Question 3 (Fast Hadamard transform.)
An important result in the history of signal processing was the fast Fourier transform (FFT), which computes the discrete Fourier transform (DFT) of a length-N vector, $|\psi\rangle$, in runtime proportional to $N\log_2(N) = Nn$, instead of N^2 for the DFT. In this question, you will derive an analogous fast Hadamard transform (FHT) that computes the discrete Hadamard transform (DHT) of $|\psi\rangle$, and show a similar computational improvement.
(a) Consider H_n, a Hadamard matrix of size $N \times N$. (Recall that one of our homework problems considered H_3 of size 8×8.) Suppose that H_n is stored in memory. Explain why computing $H_n|\psi\rangle$ using conventional matrix vector multiplication will require computation proportional to N^2. (In words, the DHT requires N^2 runtime, similar to the DFT.)
(b) The Hadamard matrix can be defined recursively,

$$H_{n+1} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix},$$

where

$$H_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Given a vector $|\psi\rangle$ of length $N = 2^n$, describe a divide and conquer style algorithm that applies H_{n-1} recursively to two halves of $|\psi\rangle$, and then combines the result. Describe the basis case and recursive part of this algorithm. (It may be simplest to describe your algorithm using some combination of pseudocode and mathematical notation.)
(c) Explain why applying this algorithm will require computation proportional to $N\log_2(N)$. (In words, the FHT requires $N\log_2(N)$ runtime, similar to the FFT.)

Question 4 (Quantum phase estimation.)
In quantum phase estimation (QPE), we apply a controlled-U^n matrix to one of its eigenvalues, $|u\rangle$. Suppose that

$$U = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

(a) Show that $|u\rangle = |1\rangle$ is an eigen-vector of U. What is the corresponding eigen-value?
(b) Suppose that you run QPE. What do you expect to sample at the QPE algorithm's output? (Hint: $i = e^{i\pi/2}$.)

Question 5 (Projects.)
Briefly describe one of the other projects that were presented. What did you learn from either the presentation in class or the report?