ECE 592-100 – Signal Processing Tour of Quantum Computing

Quiz 2 – Spring 2024

February 21, 2024

Please remember to justify your answers carefully.

Last name: ____________________ First name: ____________________

Question 1 (Nyquist rate.)
In this question, you will determine the Nyquist rate (minimum sampling rate required) for the signal,

$$x(n) = \cos(10\pi t) + \cos(20\pi t).$$

(a) Determine the Nyquist rate of the first component, $x_a(n) = \cos(10\pi t)$.

Solution. $\cos(10\pi t) = \cos(2\pi F_a t)$, hence $F_a = 5$ [cycles/time], and the Nyquist rate is $2F_a = 10$ [samples/time].

(b) Determine the Nyquist rate of the second component, $x_b(n) = \cos(20\pi t)$.

Solution. $\cos(20\pi t) = \cos(2\pi F_b t)$, hence $F_b = 10$ [cycles/time], and the Nyquist rate is $2F_b = 20$ [samples/time].

(c) Given your responses to parts (a) and (b), compute the overall Nyquist rate. Make sure to justify your answer.
(If you are unsure about your answers to parts (a) and (b), you may assume that they are $F_a = 200$ and $F_b = 100$, each in units of [samples / time].)

Solution. The overall Nyquist rate must be larger than 10 [samples/time] from part a and 20 [samples/time] from part b, hence it is 20 [samples/time].

Question 2 (LTI systems.)
Consider the discrete time system, H, whose input, $x(n)$, and output, $y(n)$, obey

$$y(n) = 0.5y(n-1) + 2x(n).$$

(a) Compute the impulse response, $h(n)$.

Solution. For $n < 0$, $h(n) = 0$.
For $n = 0$, $h(0) = 0.5h(-1) + 2\delta(0) = 2$.
For $n = 1$, $h(1) = 0.5h(0) + 2\delta(1) = 1$.
For $n = 2$, $h(2) = 0.5h(1) + 2\delta(2) = 0.5$.
For $n = 3$, $h(3) = 0.5h(2) + 2\delta(3) = 0.25$.
It seems that $h(n) = 2(0.5)^n u(n)$.

(b) Compute the Fourier transform, $H(\omega)$, of the impulse response, $h(n)$.
(If you are unsure about the answer for part (a), you may assume that $h(n) = -0.5^n u(n)$.)

Solution.

$$\begin{aligned} H(\omega) &= \sum_n h(n)e^{-i\omega n} \\ &= 2\left[\sum_{n=0}^{+\infty} 0.5^n e^{-i\omega n}\right] \\ &= \frac{2}{1-0.5e^{-i\omega}}. \end{aligned}$$

For the unsure part,

$$H(\omega) = \sum_n h(n)e^{-i\omega n} = -\sum_{n=0}^{+\infty}(0.5)^n e^{-i\omega n} = -\frac{1}{1-(0.5e^{-i\omega})}.$$

(c) Consider an input, $x(n) = 3$ (it equals 3 for all n). What will the corresponding output, $y(n)$, be?
(If you are unsure about the answer for part (b), you may assume that $H(\omega) = 1 + e^{-i\omega}$.)

Solution. The DC response of our filteris $H(\omega = 0) = 2/(1-0.5e^0) = 2/0.5 = 4$. Therefore, $y(n) = x(n)H(0) = 12$. For the unsure part, $H(\omega = 0) = 1 + e^0 = 2$ and $y(n) = 6$.

Question 3 (Linearity.)
Consider the system H that satisfies, $y(n) = x(n) + 1$. Is it linear?
If you claim that the system is linear, then you must show that $\forall \alpha, \beta \in \mathbb{R}, \forall x_1, x_2$ sequences,

$$H[\alpha x_1 + \beta x_2] = \alpha H[x_1] + \beta H[x_2].$$

If you claim that the system is non-linear, then you must provide an input sequence $x(n)$ and scalar $\alpha \in \mathbb{R}$ such that $H[\alpha x(n)] \neq \alpha H[x(n)]$.

Solution. The system is non-linear. To see why, consider $x(n) = \delta(n)$ and $\alpha = 2$. Note that

$$H[\alpha x(n)] = H[2\delta(n)] = 1 + 2\delta(n)$$

while

$$\alpha H[x(n)] = 2H[\delta(n)] = 2[1 + \delta(n)] = 2 + 2\delta(n) \neq H[\alpha x(n)].$$